Skew Interface Conditions in 2D Finite Element Machine Models

Herbert De Gersem, Kay Hameyer, Thomas Weiland

Abstract— **In 2D finite element machine models, the skewing of the stator or the rotor is commonly taken into account by considering several cross-sections at different axial positions, assembled by electrical circuit relations. Since the problem size scales with the number of slices, the computational cost rises significantly. In this paper, skew is modelled more accurately and more conveniently by imposing spectral interface conditions incorporating skew factors at a circle or an arc in the air gap.**

Index terms— **Rotating machines, Skewing, Coupled problems, Finite element methods, Fourier transforms.**

I. Introduction

SKEW is applied to electrical machines in order to reduce undesirable effects such as cogging torques, higher-harmonic air-gap fields, torque ripple, vibrations and noise. The squirrel cage of an induction machine is skewed as to filter out the first significant field harmonics due to the slotting of the machine. In permanent magnet synchronous machines, commonly, cogging torques are reduced by skewing the stator slots. The skewing of a cylindrical machine induces an electrical field in the azimuthal direction. This can lead to additional currents, e.g. if the rotor bars of an induction machine are not sufficiently insulated with respect to the lamination.

The skewing destroys the typical translatory symmetry of a cylindrical machine. Commonly, multiple slices, each of them represented by a 2D finite element (FE) model, are connected in series in order to account for the skewing. This approach substantially increase the computational cost of FE machine simulations. In this paper, the skew of the machine is modelled by an interface condition incorporating analytical skew factors at a circle in the air gap. The method alleviates the need for multiple slices in linear FE machine models. In non-linear FE models, the method results in a substantial reduction of the number of slices necessary to achieve a prescribed accuracy. The method with skew interface conditions applies to all cylindrical machines types, both stator or rotor skewing and both time-harmonic and transient models.

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II. 2D FE Machine Models

The behaviour of many cylindrical electrical machines can be simulated by 2D magnetodynamic FE models coupled to electric circuits representing external excitations and loads. The mechanical motion is taken into account by computing the torque exerted on the rotor, e.g. by the Maxwell stress tensor method, and solving the motion equation. In this paper, for convenience, the magnetodynamic formulation with skew interface conditions is developed for the time-harmonic case. The transient and multi-harmonic formulations are completely analogous. The 2D magnetodynamic partial differential equation in term of the z-component A_z of the magnetic vector potential is discretized on a triangulation of a cross-section Ω_{fe} of the machine with an $x-y$-plane. For convenience, the formulation is written in terms of a flux potential $\varphi = \ell_s A_z$ with ℓ_s the length of the device or the considered slice:

$$\begin{aligned} -\frac{\partial}{\partial x}\left(\nu\frac{\partial \underline{\varphi}}{\partial x}\right) - \frac{\partial}{\partial y}\left(\nu\frac{\partial \underline{\varphi}}{\partial y}\right) + \jmath\omega\sigma\underline{\varphi} \\ = \sigma\Delta\underline{V}_{\mathrm{sol}} + \frac{N_{\mathrm{str}}\ell_s}{S_{\mathrm{str}}}\underline{I}_{\mathrm{str}} \end{aligned} \quad (1)$$

with ν the reluctivity, σ the conductivity, ω the pulsation, $\Delta\underline{V}_{\mathrm{sol}}$ the voltage drop along a solid conductor, N_{str} the number of turns, S_{str} the cross-sectional area and $\underline{I}_{\mathrm{str}}$ the current applied to a stranded conductor. Underlined symbols indicate phasor quantities. The discretization of (1) by linear triangular FE shape functions $N_i(x,y)$ yields the system of equations

$$K\underline{u} = \underline{f} \quad (2)$$

with $\underline{u}_j$ the FE degrees of freedom for $\underline{\varphi}$,

$$\begin{aligned} K_{ij} &= \int_{\Omega_{\mathrm{fe}}} \left(\nu\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \nu\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} + \jmath\omega\sigma N_i N_j\right) \mathrm{d}\Omega \; ; \\ f_i &= \int_{\Omega_{\mathrm{fe}}} \frac{N_{\mathrm{str}}\ell_s}{S_{\mathrm{str}}} N_i \,\mathrm{d}\Omega\, \underline{I}_{\mathrm{str}} + \int_{\Omega_{\mathrm{fe}}} \sigma N_i \,\mathrm{d}\Omega\, \Delta\underline{V}_{\mathrm{sol}} \; . \end{aligned} \quad (3)$$

The voltage drops $\Delta\underline{V}_{\mathrm{sol}}$ and currents $\underline{I}_{\mathrm{str}}$ are considered as unknowns in an electric circuit model which includes the external sources and impedances. The field-circuit coupling scheme adds a few algebraic equations to the FE system as described in [1].

III. Multi-Slice 2D Machine Models

A 2D FE model as represented by (2) is justified by the translatory symmetry which is typical for many electrical machines. If, however, the stator slots, the rotor slots or the

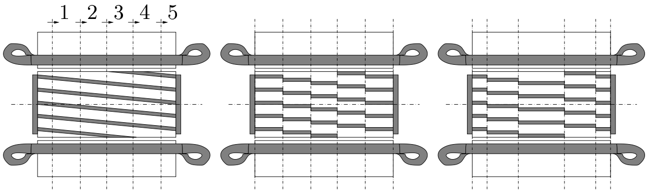

Fig. 1. FE machine models of a cylindrical machine with skewed rotor with multiple slices distributed equidistantly or according to Gauss points.

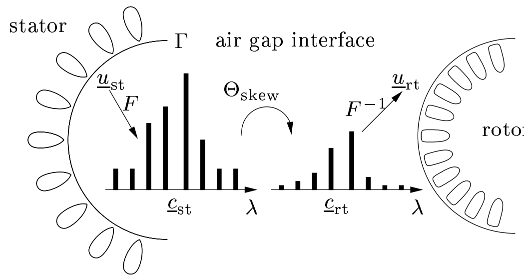

Fig. 2. Scheme of the method with skew interface conditions.

permanent magnet parts are skewed along the axis of the device, the accuracy of such model may become troublesome (Fig. 1). Several approaches to overcome these problem have been reported in literature. The most popular method is the multi-slice technique originally developed by Piriou and Razek [2]. The machine with axial length ℓ_z is cut into $n_{\rm sl}$ slices of length $\ell_s = \ell_z/n_{\rm sl}$, each of them being represented by a conventional 2D FE model. The currents through the stator windings and the rotor bars are forced to be the same in each of the slices by the field-circuit coupling scheme (Fig. 1b). Considering several slices can be done for the skewed part only [2] or for the entire device [3], [4], [5]. In [4], it has been shown that a distribution of the slices according to a Gauss point distribution offers a better convergence of the skew discretization error compared to the classical equidistant distribution (Fig. 1c). Skewed rotor bars in induction machines give rise to interbar currents which are not taken into account by the multi-slice models so far. They are considered in the electrical network by additional resistors put in between the individual slices [6]. The skewing of one of the motor parts also causes the magnetic vector potential not to be aligned with the axis of the machine as is assumed by 2D FE machine models. This effect is commonly neglected in 2D multi-slice models. In [7], the possibility of explicitly assigning the direction in which the magnetic vector potential is assumed to be invariant, along the axis or along the skewed conductors is studied. Dependent on the operation mode of the machine (locked rotor, no-load, load), a 2D, a 2D-3D or a full 3D model can be chosen. All multi-slice techniques have in common that the computation time increase, in the optimal case, linearly with the number of slices considered in the model. A combination of FE models with an analytical model accounting for the skew is proposed in [5]. The approach solves the multiple slices separately, which avoids the scaling of the FE model size with the number of slices. The skewing of the machine is taken into account by appropriately combining the coupled inductance matrices extracted from the FE solutions. In this paper, fully coupled FE machine models are considered. The number of slices required to achieve a sufficiently accurate description of the skewing is reduced by a particular kind of interface conditions applied to a circle in the air gap.

IV. Skew Interface Conditions

Analytical machine models are constructed based on rotating field theory (Fig. 2). The magnetic field generated by the stator windings in the air gap is written as a sum of rotating waves. The skewing of e.g. the rotor is taken into account by skew factors applied to the rotating-field coefficients [8]. The skew factors act as a filter to the air-gap field. The application of such skew factors in a post-processing step to a FE solution does not account for stator-slot aliasing effects [5]. Here, the skew factors are introduced in the FE model itself. A fully coupled system of equations is assembled. It would also be possible, albeit inefficient, to iterate between two FE models, one for the stator and one for the rotor, while applying skew interface conditions in order to propagate the air gap field between both FE model parts.

The stator and rotor model parts are disconnected at a circle or, in case of partial machine models, an arc $\Gamma_{\rm st} = \Gamma_{\rm rt}$ in the air gap. At this interface, two independent vectors of FE degrees of freedom, $\underline{u}_{\rm st}$ and $\underline{u}_{\rm rt}$ are defined. The remaining FE degrees of freedom in both stator and rotor, are gathered in $\underline{u}_{\diamondsuit}$. A 2D FE machine model is constructed as described in Section II. Note that this FE system consists of two non-coupled blocks of equations. The vector of degrees of freedom is

$$\underline{u} = \left[\begin{array}{ccc} \underline{u}_{\diamondsuit} & \underline{u}_{\rm st} & \underline{u}_{\rm rt} \end{array} \right]^T \tag{4}$$

and contains superfluous degrees of freedom. The FE procedure so far, assumes homogeneous Neumann boundary conditions at $\Gamma_{\rm st}$ and $\Gamma_{\rm rt}$. The skewing of the rotor with respect to the stator is introduced as an interface condition between $\Gamma_{\rm st}$ and $\Gamma_{\rm rt}$. The magnetic field at $\Gamma_{\rm rt}$ is obtained by averaging the field at $\Gamma_{\rm st}$. The continuous *skew interface conditions* read

$$\underline{\varphi}_{\rm rt}\left(\theta\right) = \frac{1}{\Delta\theta_{\rm skew}} \int_{-\Delta\theta_{\rm skew}/2}^{+\Delta\theta_{\rm skew}/2} \underline{\varphi}_{\rm st}\left(\theta+\psi\right)\, {\rm d}\psi \ , \tag{5}$$

where $\Delta\theta_{\rm skew}$ is the skew angle. This expression is discretized at $\Gamma_{\rm st}$ and $\Gamma_{\rm rt}$ following the Galerkin procedure. The potentials $\underline{\varphi}_{\rm rt}$ and $\underline{\varphi}_{\rm st}$ are written in terms of the FE shape functions $N{\rm rt},q$ and $N{\rm st},w$ restricted to $\Gamma_{\rm rt}$ and $\Gamma_{\rm st}$ respectively:

$$\underline{\varphi}_{\rm rt} = \sum_q \underline{u}_{{\rm rt},q} N_{{\rm rt},q}(\theta) \ ; \tag{6}$$

$$\underline{\varphi}_{\rm st} = \sum_w \underline{u}_{{\rm st},w} N_{{\rm st},w}(\theta) \ . \tag{7}$$

The skew interface condition (5) is weighted by the shape functions of one of both sides, e.g. by $N_{\mathrm{rt},p}(\theta)$, yielding

$$M\underline{u}_{\mathrm{rt}} = S\underline{u}_{\mathrm{st}} \tag{8}$$

with

$$\begin{aligned} M_{pq} &= \int_0^{2\pi} N_{\mathrm{rt},p}(\theta) N_{\mathrm{rt},q}(\theta)\,\mathrm{d}\theta\ ; \quad (9)\\ S_{pw} &= \int_0^{2\pi} \int_{-\Delta\theta_{\mathrm{skew}}/2}^{+\Delta\theta_{\mathrm{skew}}/2} \frac{N_{\mathrm{rt},p}(\theta) N_{\mathrm{st},w}(\theta+\psi)}{\Delta\theta_{\mathrm{skew}}}\,\mathrm{d}\psi\,\mathrm{d}\theta\ . \end{aligned}$$

The discretrized interface conditions serve as additional equations for the FE degrees of freedom, $G\underline{u} = 0$, with

$$G = \left[\begin{array}{ccc} 0 & S & -M \end{array}\right] \tag{10}$$

and are inserted as constraint equations in a saddle-point model

$$\left[\begin{array}{cc} K & G^T \\ G & 0 \end{array}\right] \left[\begin{array}{c} \underline{u} \\ \underline{v} \end{array}\right] = \left[\begin{array}{c} \underline{f} \\ 0 \end{array}\right] \tag{11}$$

with $\underline{v}$ a set of Lagrange multipliers. The terms $G^T\underline{v}$ represent the tangential magnetic field strengths weighted by the FE shape functions $N_{\mathrm{rt},p}(\theta)$ at Γ_{rt} and Γ_{st}.

The construction of the matrices M and S may be cumbersome if the FE meshes at Γ_{rt} and Γ_{st} do not match, are not equidistant or do not add up to the skew angle $\Delta\theta_{\mathrm{skew}}$. Here, instead, the skew interface conditions are discretized by a spectral technique. The magnetic fields at Γ_{st} and Γ_{rt} are related to the Fourier coefficients

$$\begin{aligned} \underline{c}_{\mathrm{st}} &= F\underline{u}_{\mathrm{st}}\ ; \quad (12)\\ \underline{c}_{\mathrm{rt}} &= F\underline{u}_{\mathrm{rt}} \quad (13) \end{aligned}$$

with F the discrete Fourier transformation (Fig. 2). In the spectral domain, the skew interface conditions (5) read $\underline{c}_{\mathrm{rt},\lambda} = \Lambda_{skew,\lambda\lambda}\,\underline{c}_{\mathrm{st},\lambda}$ with

$$\Lambda_{skew,\lambda\lambda} = \frac{\sin\left(\frac{\lambda\Delta\theta_{\mathrm{skew}}}{2}\right)}{\frac{\lambda\Delta\theta_{\mathrm{skew}}}{2}} \tag{14}$$

the *skew factor* for the component with harmonic order λ, analogous to the rotating field theory. The skew interface conditions are represented by the constraint equation $B\underline{u} = 0$ with

$$B = \left[\begin{array}{ccc} 0 & \Lambda_{\mathrm{skew}}F & -F \end{array}\right] \tag{15}$$

and constitute, together with the FE model, the saddle-point problem

$$\left[\begin{array}{cc} K & B^H \\ B & 0 \end{array}\right] \left[\begin{array}{c} \underline{u} \\ \underline{\xi} \end{array}\right] = \left[\begin{array}{c} \underline{f} \\ 0 \end{array}\right]\ , \tag{16}$$

where $\underline{\xi}$ is a vector of Lagrange multipliers.

Solving (11) or (16) is more expensive than solving a 2D FE model without skew interface conditions which is the main disadvantage of this formulation. If the FE mesh is equidistant at Γ_{st} and Γ_{rt}, it is recommended not to construct B and B^H, but to apply (inverse) Fast Fourier Transforms and explicit scaling operations for F^H, F and Λ_{skew}. Based on such matrix-free techniques, powerful domain-decomposition-type iterative solution approaches are proposed in [9]. The increase of the simulation time is typically only 10%. This additional complexity is acceptable since skew interface conditions are expected to reduce the number of slices required to obtain a prescribed simulation accuracy. This technique with skew interface conditions can easily be combined with analytical air gap element techniques [10], [11], air gap flux splitting approaches [12] and sliding surface methods [13].

V. Skew Discretization Error

The *relative skew discretization errors* are defined by the relative differences between the discrete skew factors introduced by the multi-slice model and the exact skew factors. In [4], it was shown that distributing the slices according to a Gaussian integration scheme results in a better relative skew discretization error compared to the case where the slices are distributed equidistantly. The skew discretization errors for the skew interface conditions developed here, do not depend on the number of slices. They are only determined by the FFTs and hence, by the number of points at the skew interface.

The skew discretization errors as defined above, however, are only indications of the error introduced by multi-slice techniques. A better qualitative comparison of skew modelling techniques is described in the following. Taking multiple slices corresponds to discretizing the model in the z-direction. The classical multi-slice technique considers n_{sl} equidistantly distributed slices, each of them neglecting the skew, which corresponds to n_{sl} constant FEs along the axis of the device. The distribution of n_{sl} slices following a Gauss distribution corresponds to the use of a spectral discretization technique with the Legendre polynomials up to degree n_{sl} acting as shape functions [14]. It is known that for spectral elements, the discretization error decays at exponential rather than at polynomial rate in case of smooth excitations and geometries. The geometry and excitation of a typical electrical machine indeed vary smoothly with respect to the z-direction. The technique with skew interface conditions is also a spectral element technique, but with trigonometric shape functions. For linear models, it is exact up to the FFT discretization error. A single-slice FE model with skew interface conditions obviously does not account for axial variations of the ferromagnetic saturation. Skew interface conditions can be applied in combination with the multi-slice approach. Then, FE models with skew interface conditions are expected to achieve at least the same accuracy as the approach with Gauss points, also for non-linear models. In practice, the same accuracy is already obtained with a substantially smaller number of slices than for the other approaches.

VI. Application

The 2D formulations with multiple slices and with skew interface conditions are applied to a time-harmonic induction machine model (Fig. 3). The 4-pole 45 kW machine has 48 stator slots and 36 rotor slots. The machine is sim-

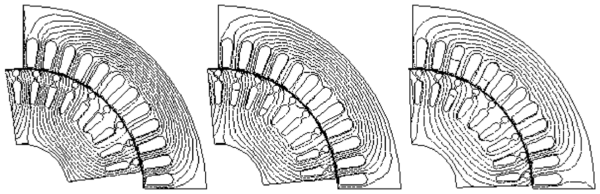

Fig. 3. 3-slice model of an induction machine.

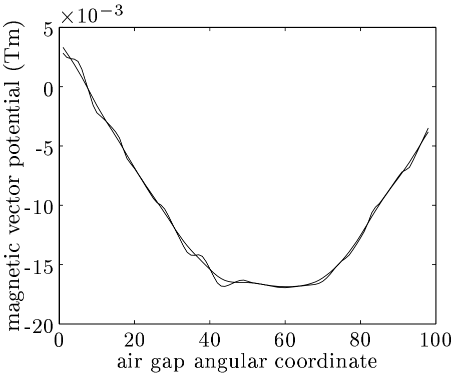

Fig. 4. Magnetic vector potential distribution at the stator side Γ_{st} and the rotor side Γ_{rt} of the skew interface contour in the air gap.

ulated at full load and at its nominal speed of 1472.7 rpm. Three formulations are compared: the classical technique with multiple slices equidistantly distributed along the machine's axis, the improved multi-slice technique distributed slices according to Gauss points and the new technique applying skew factors at an interface in the air gap. The machine has closed rotor slots at which substantial ferromagnetic saturation is observed. The stator end-windings, the rotor rings and the three-phase voltage source are modelled by an external electrical circuit. The field-circuit coupling scheme is also used to define the appropriate connections between the stator windings and rotor bars of different slices. The magnetic field distributions at the stator side Γ_{st} and the rotor side Γ_{rt} of the skew interface contour are shown in Fig. 4. The convergence of the error is compared for the torque (Fig. 5). The distribution of the slices according to Gauss points offers a better convergence compared to the equidistant distribution. The approach with skew interface conditions provides the best convergence and already accounts for skewing when only one slice is considered.

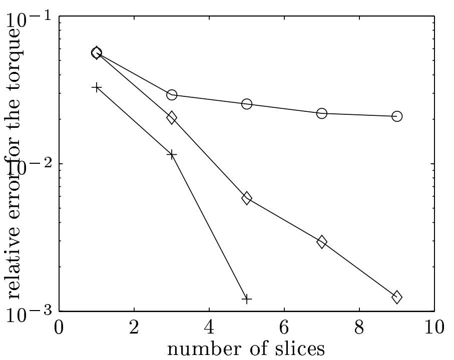

Fig. 5. Convergence of the error on the torque for the classical multi-slice technique ($\circ$), the multi-slice technique with slices distributed according to Gauss points ($\Diamond$) and the multi-slice technique combined with skew interface conditions ($+$).

VII. Conclusions

Skew can be taken into account in 2D FE machine models by interface conditions with skew factors applied at a circle in the air gap. The new method offers reduced computation times and increased modelling flexibility.

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