Electromagnetic force densities in the Finite Element context

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I. Introduction

THE idea behind the definition of electromagnetic (EM) forces is well known : they are determined by the variation of the EM energy when a configuration parameter (e.g. the position of one node) is modified, keeping the EM field constant. Although this definition, which is basically a partial derivative in a carefully defined parameter set, is rather simple to state, the different steps to its implementation in a finite element (FE) programme remain quite obscure and uncertain. Moreover, there exist two distinct families of EM force formulae. The ones belonging to the first family are based on the *Maxwell stress tensor* while the other ones stem from the application of the *virtual work principle*. This distinction is another aspect that makes the situation unclear.

A survey into the literature shows that the expression of the Maxwell stress tensor (See e.g. [1]) is generally obtained by algebro-differential operations starting from the Maxwell equations. The virtual work principle, on the other hand, relies more clearly on energy concepts but the formulae are obtained by a cumbersome roundabout way involving the jacobian matrix of coordinate transformation [2], [3], [6], [4], [5]. In both cases, coordinates are used and the fundamental thermodynamical concepts are buried into an overwhelming algebra. The issue has also been treated in a coordinate-free manner in [7], [8] but at the expense of resorting to the unusual mathematical framework of differential geometry.

Another observation is that all these approaches disregard the role played by the underlying matter. They assume *a priori* a specific expression for the magnetic constitutive law but fail to ask the fundamental question : How is this law affected by deformation ? In the end, the conditions of applicability of the classical formulae are vague and hardly interpretable in the context of a new material for instance. Consequently, the blind test consisting in the numerical confrontation of different formulae has been quite a popular game [9], [10], [11], [12].

This paper is organised as follows. In section II, the thermodynamic definition of the electromechanical coupling is recalled and carefully placed in the variational context [13]. Duality aspects are considered. In section III, the definition is applied to the particular case of an infinitesimal box. In presence of a linear material, this leads straightforwardly to the expression of the Maxwell stress tensor as well as to that of classical formulae obtained by the virtual work principle, thus making the link between them obvious. The derivation being concise, the applicability conditions show up much more intelligibly and can be reviewed. Dual magnetic formulations are considered. In section IV, by appying the approach to permanent magnets and nonlinear materials, it is shown how magnetic constitutive laws are affected by the deformation of the underlying space, i.e. how the distinction between $h-$like ($1-$forms) and $b-$like fields ($2-$forms), which is irrelevant in the expression of the *electromagnetic* constitutive laws, becomes essential when the *electromechanical* coupling are considered. In section V, finally, the tensor nature of the EM force density and that of the Maxwell stress tensor are analysed under the light of the uncompletely antisymmetrised tensors defined in [14].

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II. EM forces in the variational context

Let $\rho^\Psi(b,u)$ be the *energy density* functional of an electromechanical system Ω. It depends on two independent variables : the induction field b and a displacement field u. If the problem is posed in terms of the field h, instead of b, the available functional is the *coenergy density* functional $\rho^\Phi(h,u)$ and the energy density functional is now defined by [13]

$$\rho^\Psi(b,u) = \min_h \left\{ h \cdot b - \rho^\Phi(h,u) \right\}. \qquad (1)$$

Each coil is associated with a pair of dof's : a flux ϕ and a current I. Those fluxes and currents are *global parameters* that play a part in the definition of the fields : b_ϕ is a divergence-free induction field that matches the fluxes ϕ in the coils while h_I is a magnetic field that matches the currents I. On the other hand, moving parts of the system Ω are associated with global parameters X and a continuous displacement field that matches those displacements is noted u_X.

This system is acted upon by external agents that are able to impose either the flux ϕ or the current I in the coils, and also to impose the displacement X of the moving parts. Let us first consider that all coils are flux driven and that all moving pieces are at known positions. The thermodynamical state function of the system is the *energy* function

$$\Psi(\phi, X) = \min_{b_\phi, u_X} \int_{\Omega(u_X)} \rho^\Psi(b_\phi, u_X). \qquad (2)$$

of the controle variables of the system, ϕ and X. The minimisation ensures that the system is stationarised, i.e. that physical laws (Ampere law and equilibrium) are verified. The formulation in h, still with imposed fluxes, writes

$$\Psi(\phi, X) = \min_{h_I, I, u_X} \int_{\Omega(u_X)} \left\{ h_I \cdot b_\phi - \rho^\Phi(h_I, u_X) \right\} \qquad (3)$$

and one has for both formulations

$$I(\phi) = \partial_\phi \Psi \quad , \quad F(X) = \partial_X \Psi \qquad (4)$$

where the partial derivatives imply that the derivation with respect to one variable is performed keeping the other variables constant. This is our only definition of an *electromagnetic force*.

III. CLASSICAL MAXWELL STRESS TENSOR

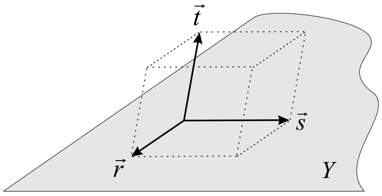

Fig. 1. Parallelepiped box

Assuming an euclidean underlying space, it is enough, so as to avoid using coordinates, to work in the framework of vector analysis. This renders the following derivation fairly concise, though perfectly rigorous and general. The only conceptual addition brought to the theory of vector analysis is the distinction, amongst the vector fields, between those that integrate naturally over a curve (the $1-$forms) and those that integrate over a surface (the $2-$forms).

A. *Formulation in b*

Let us consider the *parallelepipedic* box defined by the vectors $\vec{r}$, $\vec{s}$ and $\vec{t}$, Fig. 1. The box is taken small enough to have a uniform induction field $\vec{b}$ inside. The fluxes across the facets of the parallelipided are by definition

$$\begin{pmatrix} \phi_{st} \\ \phi_{tr} \\ \phi_{rs} \end{pmatrix} = \begin{pmatrix} \vec{s} \times \vec{t} \\ \vec{t} \times \vec{r} \\ \vec{r} \times \vec{s} \end{pmatrix} \cdot \vec{b}. \tag{5}$$

Since the 3x3 matrix in (5) admits as inverse

$$\begin{pmatrix} \vec{s} \times \vec{t} \\ \vec{t} \times \vec{r} \\ \vec{r} \times \vec{s} \end{pmatrix}^{-1} = \frac{1}{V} \begin{pmatrix} \vec{r} & \vec{s} & \vec{t} \end{pmatrix}, \tag{6}$$

where $V = (\vec{r} \times \vec{s}) \cdot \vec{t}$ is the volume of the box, the induction field is expressed as a function of the fluxes by

$$\vec{b} = \left(\frac{\vec{r}}{V} \phi_{st} + \frac{\vec{s}}{V} \phi_{tr} + \frac{\vec{t}}{V} \phi_{rs} \right), \tag{7}$$

the factors $\vec{r}/V$, $\vec{s}/V$ and $\vec{t}/V$ being facet shape functions for the parallipipedic region with uniform field. If the box is made of a material of which the constitutive law is $b = \mu h$, with a constant magnetic permeability μ, the magnetic energy in the box is

$$\Psi = V \frac{|\vec{b}|^2}{2\mu} = \frac{1}{2\mu V} \left| \vec{r}\phi_{st} + \vec{s}\phi_{tr} + \vec{t}\phi_{rs} \right|^2. \tag{8}$$

The box is now deformed by perturbating the vector $\vec{t}$ by an increment $\delta\vec{t}$, leaving $\vec{r}$ and $\vec{s}$ unchanged. The variation of the energy (8) writes

$$\begin{aligned} \delta\Psi &= \frac{1}{\mu V} \left(\vec{r}\phi_{st} + \vec{s}\phi_{tr} + \vec{t}\phi_{rs} \right) \cdot \delta\vec{t}\phi_{rs} \\ &- \frac{1}{2\mu V^2} \left| \vec{r}\phi_{st} + \vec{s}\phi_{tr} + \vec{t}\phi_{rs} \right|^2 \delta V. \end{aligned} \tag{9}$$

where we are now allowed to substitute back for $\vec{b}$ using (7)

$$\delta\Psi = \frac{\phi_{rs}}{\mu} \vec{b} \cdot \delta\vec{t} - \frac{|\vec{b}|^2}{2\mu} \delta V. \tag{10}$$

Since $\phi_{rs} = (\vec{r} \times \vec{s}) \cdot \vec{b}$ and $\delta V = (\vec{r} \times \vec{s}) \cdot \delta\vec{t}$, one has

$$\delta\Psi = (\vec{r} \times \vec{s})^i \, \sigma_{ij} \, \delta t^j \quad , \quad \sigma_{ij} = \frac{1}{\mu} \left(b_i b_j - \frac{|\vec{b}|^2}{2} \delta_{ij} \right) \tag{11}$$

which is the classical definition of the *Maxwell stress tensor*.

In case of the formulation in h, the magnetic field inside the box is represented as

$$\vec{h} = \left(\frac{\vec{s} \times \vec{t}}{V} I_r + \frac{\vec{t} \times \vec{r}}{V} I_s + \frac{\vec{r} \times \vec{s}}{V} I_t \right). \tag{12}$$

where the factors $\frac{\vec{s} \times \vec{t}}{V}, \ldots$ are edge shape function for the parallipipedic region with uniform field. One sees here how much the representation of a $1-$form (12) and the representation of a $2-$form (7) are different from each other. Even if both are vectors, their behaviour under a perturbation $\delta\vec{t}$ will be quite different. This has important implications on the electromechanical behaviour of a permanent magnet materials as will be shown in the full paper.

On the other hand, the formulae for local EM forces stemming from the application of the Virtual work principle [3], [6] are found back by applying the same procedure to a *tetrahedral* box underlied by the same vectors $\vec{r}$, $\vec{s}$ and $\vec{t}$. This will be detailed in the full paper as well.

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