

# Computation of electromagnetic force densities : Maxwell stress tensor vs virtual work principle \*

François Henrotte<sup>†</sup>, Kay Hameyer  
Katholieke Universiteit Leuven, Dep. ESAT/ELECTA,  
Kasteelpark Arenberg 10, B-3001 Leuven, Belgium.

## Abstract

In this paper, the thermodynamic statement of the electro-mechanical coupling is recalled and carefully placed in the variational context. A calculation of electromagnetic force density is proposed that leads to the expression of the Maxwell stress tensor as well as to that of classical local EM forces formulae obtained by the virtual work principle, making thus the link between them obvious. The calculation being concise, the applicability conditions of the EM force formulae show up more clearly. The proposed demonstration provides in addition a theoretical framework to investigate how magnetic constitutive laws are affected by the deformation of the underlying space.

*Keywords:* electromagnetic forces, finite element, Maxwell stress tensor, virtual work principle

*AMS classification:*

## 1 Introduction

The idea behind the definition of electromagnetic (EM) forces is well known : they are given by the variation of the EM energy when a configuration parameter (e.g. the position of one node) is modified, keeping the EM field constant. Although this definition, which is basically a partial derivative in a properly defined variable set, is rather simple to state, the different steps to its implementation in a finite element (FE) programme remain quite obscure and uncertain. Moreover, there exist two distinct families of EM force formulae, depending on whether they are based on the *Maxwell stress tensor* or on the application of the *virtual work principle*. This distinction is another aspect that makes the situation unclear.

A look into the literature shows that the expression of the Maxwell stress tensor (See e.g. [20]) is generally obtained by algebro-differential operations starting from the Maxwell equations. The virtual work principle, on the other hand, relies more clearly on energy concepts but the formulae are obtained by a cumbersome roundabout way involving the jacobian matrix of coordinate transformation [2, 14, 10, 15, 17]. In both cases, coordinates are used (at least at some places) and the fundamental thermodynamic concepts are buried into an overwhelming algebra. The issue has also been treated in a coordinate-free manner in [3, 4] but at the expense of resorting to the more involved mathematical framework of differential geometry.

Another observation is that all these approaches disregard the role played by the underlying matter. They assume *a priori* a specific expression for the magnetic constitutive law but fail to ask the fundamental question : How is this law affected by deformation ? In the end, the conditions of applicability of the

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<sup>†</sup>Corresponding author. E-mail: Francois.Henrotte@esat.kuleuven.ac.be

classical formulae are unclear, and hardly interpretable in the context of a new material for instance. The blind test consisting in the numerical confrontation of different formulae has therefore been quite a popular game [11, 13, 16, 18].

The purpose of this paper is to demonstrate how one can differentiate energy with respect to a configuration variable and keep the electromagnetic field constant *in some sense*, i.e. consistently with the thermodynamics of the electro-mechanical coupling and with the requirements of the variational and finite element contexts. The paper is organised as follows. The needed mathematical framework is presented in section 2. In section 3, the thermodynamic definition of the electro-mechanical coupling is recalled and carefully placed in the variational context. Duality aspects are considered. In section 4, the definition is applied to a deforming infinitesimal box in the case of a non-magnetic material. It leads to the well-known expression of the Maxwell stress tensor. In addition, it is shown that the induction field  $\vec{b}$  and the magnetic field  $\vec{h}$  have quite a different behaviour under deformation, although they are both vector fields. The direct between Maxwell stress tensor and the virtual work principle is shown in section 5. The calculation of the two previous sections being concise and self-supporting, the *applicability conditions* of the final formulae show up more clearly. They are reviewed in section 6. Two examples of magnetic materials are considered in section 7.

## 2 An intrinsic Euclidean continuous medium framework

We wish to present an intrinsic theory. In the particular case of an Euclidean 3D space, *vector analysis* provides the relevant intrinsic notions (e.g. vector, cross product, dot product, gradient, ...). Some basic notions of *tensor analysis* (e.g. tensor product, trace, ...) are also added in order to complete the mathematical framework.

Tensors are commonly seen as arrays of scalar numbers obeying the product rule

$$T^{m \times n} \times T^{n \times p} \mapsto T^{m \times p} : [AB]_{ik} = [A]_{ij}[B]_{jk} \quad (2.1)$$

where  $T^{m \times n}$  is the  $m \times n$  dimensional vector space of tensors with  $m$  lines and  $n$  columns, square brackets denote the components of a tensor and implicit summation is assumed. Let us note  $E \equiv T^{3 \times 1}$  and  $E^* \equiv T^{1 \times 3}$ , which are two dual three-dimensional vector spaces. By application of (2.1), the products of vectors and dual vectors are defined as

$$\begin{aligned} E^* \times E &\mapsto R & : \vec{u}^* \vec{v} &= u_i^* v_i \\ E \times E^* &\mapsto E^2 & : [\vec{u} \vec{v}^*]_{ij} &= u_i v_j^* \end{aligned} \quad (2.2)$$

where  $R \equiv T^{1 \times 1}$  is the space of real numbers and  $E^2 \equiv T^{3 \times 3}$  is the space of  $3 \times 3$  tensors. The  $\star$  operator is defined by

$$\begin{aligned} \star : R &\mapsto R & : \lambda^* &= \lambda \\ \star : E &\mapsto E^* & : u_i^* &= u_i \\ \star : E^2 &\mapsto E^2 & : [A^*]_{ij} &= [A]_{ji} \end{aligned} \quad (2.3)$$

It is similar to the transpose operator of tensor analysis. In particular, it is such that  $x^{**} = x$  and  $(xy)^* = y^* x^*$ ,  $\forall x, y$ . The *dot product of two vectors* is defined by

$$\cdot : E \times E \mapsto R : \vec{b} \cdot \vec{h} = b_i h_i. \quad (2.4)$$

It can also be seen as the combined application of the  $\star$  operator (2.3) and the tensor product (2.1) :  $\vec{b} \cdot \vec{h} = \vec{b}^* \vec{h} = \vec{h}^* \vec{b}$ . We will constantly switch between both representations of the dot product. The *dot product of two tensors* is defined by

$$\cdot : E^2 \times E^2 \mapsto R : A \cdot B = [A]_{ij}[B]_{ij} \quad (2.5)$$

so that  $\vec{u}^* A \vec{v} = A \cdot (\vec{u} \vec{v}^*)$ . Finally,  $E$  is also equipped with an antisymmetric *cross product*

$$\times : E \times E \mapsto E : (\vec{u} \times \vec{v})_k = \epsilon_{ijk} u_i v_j \quad (2.6)$$

where  $\epsilon_{ijk}$  is the Levi-Civita symbol.

Let  $M$  be a unit cube, with coordinates  $\{\alpha, \beta, \gamma\}$ , of which each point represents a material particle. If  $\vec{r}$ ,  $\vec{s}$  and  $\vec{t}$  are three linearly independent vectors of  $E$ , the 1-1 map

$$\varphi : M \mapsto E : \vec{x} = \alpha \vec{r} + \beta \vec{s} + \gamma \vec{t} \quad (2.7)$$

determines in  $E$  a parallelepiped region of volume

$$V = (\vec{r} \times \vec{s})^* \vec{t} = (\vec{s} \times \vec{t})^* \vec{r} = (\vec{t} \times \vec{r})^* \vec{s}. \quad (2.8)$$

The following tensor product is now considered

$$\frac{1}{V} \begin{bmatrix} (\vec{s} \times \vec{t})^* \\ (\vec{t} \times \vec{r})^* \\ (\vec{r} \times \vec{s})^* \end{bmatrix} [\vec{r} \quad \vec{s} \quad \vec{t}] = I \quad (2.9)$$

where  $I$  is the identity tensor. The two bracketed terms at the l.h.s. are  $3 \times 3$  tensors, but they can as well be seen as a  $3 \times 1$  tensor of dual vectors and a  $1 \times 3$  tensor of vectors. Applying now the tensor product rules (2.1), the arising terms are products of vectors and dual vectors, which are defined by (2.2). The equality (2.9) is a consequence of (2.6) and (2.8). It shows that the tensors at the l.h.s. are inverse of each other. Therefore, they commute and one has as well

$$\vec{r} \frac{(\vec{s} \times \vec{t})^*}{V} + \vec{s} \frac{(\vec{t} \times \vec{r})^*}{V} + \vec{t} \frac{(\vec{r} \times \vec{s})^*}{V} = I. \quad (2.10)$$

By right-multiplying both sides of (2.10) with  $\vec{x}$ , and identifying with (2.7), one finds the expression of the inverse map  $\varphi^{-1} : E \mapsto M$ :

$$\alpha = \frac{(\vec{s} \times \vec{t})^* \vec{x}}{V}, \quad \beta = \frac{(\vec{t} \times \vec{r})^* \vec{x}}{V}, \quad \gamma = \frac{(\vec{r} \times \vec{s})^* \vec{x}}{V}. \quad (2.11)$$

The *gradient* operator is defined by  $\vec{\nabla} \vec{x}^* = I$ , Hence,

$$\vec{\nabla} \alpha = \frac{(\vec{s} \times \vec{t})^*}{V}, \quad \vec{\nabla} \beta = \frac{(\vec{t} \times \vec{r})^*}{V}, \quad \vec{\nabla} \gamma = \frac{(\vec{r} \times \vec{s})^*}{V}. \quad (2.12)$$

The notion we wish to represent now is that of a *homogeneous deformation*. Therefore, the vectors  $\vec{r}$ ,  $\vec{s}$  and  $\vec{t}$  are perturbed by  $\delta \vec{r}$ ,  $\delta \vec{s}$  and  $\delta \vec{t}$  respectively. The perturbation vectors are the configuration parameters that will be generically noted  $X$  in the next section. The *displacement*  $\vec{u}_X \in E$  of one material point with coordinate  $\{\alpha, \beta, \gamma\}$  is the vector

$$\vec{u}_X = \alpha \delta \vec{r} + \beta \delta \vec{s} + \gamma \delta \vec{t} \quad (2.13)$$

and the *gradient of the displacement*,  $(\vec{\nabla} \vec{u}_X)^* \in E^2$ , is the tensor

$$(\vec{\nabla} \vec{u}_X)^* = \frac{1}{V} (\delta \vec{r} (\vec{s} \times \vec{t})^* + \delta \vec{s} (\vec{t} \times \vec{r})^* + \delta \vec{t} (\vec{r} \times \vec{s})^*). \quad (2.14)$$

The 9 dimensional vector space  $E^2$  can be decomposed into 3 irreducible subgroups [19, 9]. For any  $A \in E^2$ , one distinguishes the 1-dimensional subgroup  $tr(A) \frac{I}{3}$  where  $tr(A) = [A]_{ii}$  is the *trace* of  $A$ , the 5-dimensional subgroup  $\hat{A} = \frac{A+A^*}{2} - tr(A) \frac{I}{3}$  (symmetric and traceless) called *deviator* and the 3-dimensional subgroup of *infinitesimal rotations*, which is the antisymmetric part  $\frac{A-A^*}{2}$  of the tensor.

### 3 EM forces in the variational context

Let  $\rho^\Psi(\vec{b}, \vec{u})$  be the *energy density* of an electro-mechanical system  $\Omega$ . It depends on two independent variables: the induction field  $\vec{b}$  and a displacement field  $\vec{u}$ . If the problem is posed in terms of the unknown field  $\vec{h}$ , instead of  $\vec{b}$ , the available functional is the *co-energy density*  $\rho^\Phi(\vec{h}, \vec{u})$  and the energy density is defined by [7, 5]

$$\rho^\Psi(\vec{b}, \vec{u}) = \min_{\vec{h}} \left\{ \vec{h} \cdot \vec{b} - \rho^\Phi(\vec{h}, \vec{u}) \right\}. \quad (3.15)$$

Each coil in the system is associated with a pair of degrees of freedom: a flux  $\phi$  and a current  $I$ . Those fluxes and currents are *global parameters* that play a part in the definition of the fields:  $\vec{b}_\phi$  is a divergence-free induction field that matches the fluxes  $\phi$  in the coils while  $\vec{h}_I$  is a magnetic field that matches the currents  $I$ . On the other hand, moving parts of the system  $\Omega$  are associated with global parameters  $X$  and a continuous displacement field that matches those displacements  $X$  is noted  $\vec{u}_X$ .

This system is acted upon by external agents that are able to impose either the flux  $\phi$  or the current  $I$  in the coils, and also to impose the displacement  $X$  of the moving parts. Let us first consider that all coils are flux driven and that all moving pieces are at known positions. The thermodynamic state function of the system is the *energy* function

$$\Psi(\phi, X) = \min_{\vec{b}_\phi, \vec{u}_X} \int_{\Omega(\vec{u}_X)} \rho^\Psi(\vec{b}_\phi, \vec{u}_X). \quad (3.16)$$

of the control variables of the system,  $\phi$  and  $X$ . The minimisation ensures that physical laws (Ampere law and equilibrium) are verified. The formulation in  $\vec{h}$ , still with imposed fluxes, writes

$$\Psi(\phi, X) = \min_{\vec{h}_I, I, \vec{u}_X} \int_{\Omega(\vec{u}_X)} \left\{ \vec{h}_I \cdot \vec{b}_\phi - \rho^\Phi(\vec{h}_I, \vec{u}_X) \right\}. \quad (3.17)$$

The force  $F$  is now defined, for both formulations, by

$$\delta\phi = 0 \quad \Rightarrow \quad \delta\Psi(\phi, X) = F\delta X. \quad (3.18)$$

In case of current driven coils, the formulae to use are

$$\Phi(I, X) = \min_{\vec{b}_\phi, \phi, \vec{u}_X} \int_{\Omega(\vec{u}_X)} \left\{ \vec{h}_I \cdot \vec{b}_\phi - \rho^\Psi(\vec{b}_\phi, \vec{u}_X) \right\}, \quad \Phi(I, X) = \min_{\vec{h}_I, \vec{u}_X} \int_{\Omega(\vec{u}_X)} \rho^\Phi(\vec{h}_I, \vec{u}_X) \quad (3.19)$$

respectively for the formulations in  $\vec{b}$  and  $\vec{h}$ , with now the force  $F$  defined by

$$\delta I = 0 \quad \Rightarrow \quad \delta\Phi(I, X) = -F\delta X. \quad (3.20)$$

#### 3.1 Discussion

The definitions (3.18) and (3.20) may be directly applied in practice by performing finite energy differences [1]. This requires however to solve several times the system to obtain the force.

Nothing has been said either, in this general thermodynamic statement, about the exact nature of the global force. Would  $\vec{u}_X$  describe a dilatation of a part the system,  $F$  would then be a pressure. Would  $\vec{u}_X$  be a rotation,  $F$  would be a torque. Would  $u_X$  describe the displacement of a *rigid body* in a direction  $\vec{\eta}$ ,  $F$  would be the *resultant force* acting in that direction on the body <sup>1</sup>.

<sup>1</sup>Note how tightly the notions of rigid-body movement and resultant force are linked.

Equations (3.18) and (3.20) are derived from the *total (co)energy* function of the system. They are therefore *total* forces, i.e. of mechanical *and* electromagnetic nature together. In order to define specifically *electromagnetic forces*, it is commonly assumed that it is sufficient to use an *electromagnetic (co)energy* function instead of the *total (co)energy* function. The virtue of energy, however, is that it transforms freely from one kind to another. It is not possible to isolate from other kinds of energy a certain amount of energy that would be of an electromagnetic nature and depend on the electromagnetic variables only. One can therefore not write  $\Phi(I, X) = \Phi_e(I) + \Phi_m(X)$ . What is true however is that, in the algebraic expression of an electro-mechanical (co)energy function, some terms are rather from an electromagnetic origin (even if they involve mechanical variables as well) and some others rather from a mechanical origin. In that sense, the energy functional can be split up, in a more or less significant way, into the sum of an electromagnetic and a mechanical term, i.e.  $\Phi(X, I) = \Phi_e(I, X) + \Phi_m(X, I)$ . The *electromagnetic force* is then defined as the one obtained by applying (3.18) to the first term only, but it must be kept in mind that this definition in that way are somewhat arbitrary, since the only quantity that really makes sense from the thermodynamic point of view is the total force. In any case, the important issue is to correctly identify, at the local level, the role of the mechanical variables in the electromagnetic terms, i.e. to determine, on basis of the electro-mechanical constitutive laws of the material under consideration (which can be found in treatises like [6, 12, 19]), what is the corresponding expression of the electromagnetic energy density. The latter is the only thing we need to know to compute the expression of the EM forces by applying the procedure described in the next section.

## 4 Classical Maxwell stress tensor

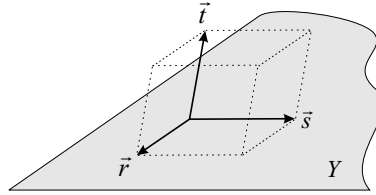


Figure 1: Parallelepiped box

### 4.1 Formulation in $b$

Let us consider the *parallelepiped* box defined by the vectors  $\vec{r}$ ,  $\vec{s}$  and  $\vec{t}$ , Fig. 1. The box is taken small enough to have a uniform induction field  $\vec{b}$  inside. The fluxes across the facets of the parallelepiped are by definition

$$\begin{bmatrix} \phi_{st} \\ \phi_{tr} \\ \phi_{rs} \end{bmatrix} = \begin{bmatrix} (\vec{s} \times \vec{t})^* \\ (\vec{t} \times \vec{r})^* \\ (\vec{r} \times \vec{s})^* \end{bmatrix} \vec{b}. \quad (4.21)$$

The induction field can be expressed as a function of the fluxes by left-multiplying (4.21) with  $[\vec{r} \ \vec{s} \ \vec{t}]/V$  and using (2.10) :

$$\vec{b} = \frac{\vec{r}}{V} \phi_{st} + \frac{\vec{s}}{V} \phi_{tr} + \frac{\vec{t}}{V} \phi_{rs}. \quad (4.22)$$

If the box is made of a non-magnetic material, i.e. of which the constitutive law is  $\vec{b} = \mu_0 \vec{h}$ , the magnetic energy in the box is

$$\Psi = V \frac{|\vec{b}|^2}{2\mu_0} = \frac{1}{2\mu_0 V} |\vec{r} \phi_{st} + \vec{s} \phi_{tr} + \vec{t} \phi_{rs}|^2. \quad (4.23)$$

The box is now deformed by perturbing the vector  $\vec{t}$  by an increment  $\delta\vec{t}$ , leaving  $\vec{r}$  and  $\vec{s}$  unchanged. The fluxes through the facets of the box are also kept constant so as to ensure the continuity of the interpolated field  $\vec{b}$  with the exterior of the box, whatever its deformation. The variation of the energy (4.23) is

$$\delta\Psi = \frac{1}{\mu_0 V} (\vec{r}\phi_{st} + \vec{s}\phi_{tr} + \vec{t}\phi_{rs}) \cdot \delta\vec{t}\phi_{rs} - \frac{1}{2\mu_0 V^2} |\vec{r}\phi_{st} + \vec{s}\phi_{tr} + \vec{t}\phi_{rs}|^2 \delta V. \quad (4.24)$$

After performing the variation, it is allowed to substitute back for  $\vec{b}$  using (4.22) so that

$$\delta\Psi = \frac{\phi_{rs}}{\mu_0} \vec{b}^* \delta\vec{t} - \frac{|\vec{b}|^2}{2\mu_0} \delta V. \quad (4.25)$$

Since  $\phi_{rs} = (\vec{r} \times \vec{s})^* \vec{b}$  and  $\delta V = (\vec{r} \times \vec{s})^* \delta\vec{t}$ , one has

$$\delta\Psi = (\vec{r} \times \vec{s})^* \sigma_M \delta\vec{t} \quad , \quad \sigma_M = \frac{1}{\mu_0} \left( \vec{b} \vec{b}^* - \frac{|\vec{b}|^2}{2} I \right) \quad (4.26)$$

which is the classical definition of the *Maxwell stress tensor of empty space*.

Before proceeding, it is worth checking that the conditions that rule the definition of EM force have been respected. In case of a problem with imposed flux, the EM force is defined by (3.18) with (3.16). As it is uniform, the induction field  $\vec{b}$  is trivially divergence-free inside the box. As it matches the flux imposed at the boundaries of the box, it connects with the right continuity to the field outside the box, whatever  $\delta\vec{t}$  which plays here the role of  $X$ . Altogether, the field  $\vec{b}$  is in the class of the  $b_\phi$  fields, i.e. it verifies Gauss law everywhere and it is always compatible with the constraints  $\phi$ . This is the consistency condition with regard to the variational principle. Ampere law is also verified provided that the particular value of induction field that is used to evaluate (4.26) is the solution of the magnetic problem (3.16), i.e. the one that minimises the functional. By identification of (4.26) with (3.18), the force we are looking for is found to be  $F = (\vec{r} \times \vec{s})^* \sigma_M$ . In case of a problem with imposed currents, the EM force is defined by (3.20) with (3.19) and it can be checked that the extra term has a zero contribution to the force. Finally, if the perturbation of the three vectors,  $\vec{r}$ ,  $\vec{s}$  and  $\vec{t}$ , is now allowed, (4.26) becomes (4.27). The Maxwell stress tensor is therefore the energy dual of the gradient of the displacement field, i.e. a true *local stress tensor* that can be used as such in the structural equations.

$$\delta\Psi = (\vec{r} \times \vec{s})^* \sigma_M \delta\vec{t} + (\vec{s} \times \vec{t})^* \sigma_M \delta\vec{r} + (\vec{t} \times \vec{r})^* \sigma_M \delta\vec{s} = V \sigma_M \cdot (\vec{\nabla} \vec{u}_X^*). \quad (4.27)$$

## 4.2 Formulation in $h$

The circulations of the magnetic field along the edges of the parallelepiped are by definition

$$\begin{bmatrix} I_r \\ I_s \\ I_t \end{bmatrix} = \begin{bmatrix} \vec{r}^* \\ \vec{s}^* \\ \vec{t}^* \end{bmatrix} \vec{h}, \quad (4.28)$$

whence, by (2.9)

$$\vec{h} = \frac{\vec{s} \times \vec{t}}{V} I_r + \frac{\vec{t} \times \vec{r}}{V} I_s + \frac{\vec{r} \times \vec{s}}{V} I_t. \quad (4.29)$$

Considering again a non-magnetic material, the magnetic co-energy in the box and its variation are

$$\Phi = V \mu_0 \frac{|\vec{h}|^2}{2} \quad , \quad \delta\Phi = \mu_0 \vec{h} \cdot \{ \vec{s} \times \delta\vec{t} I_r + \delta\vec{t} \times \vec{r} I_s \} - \mu_0 \frac{|\vec{h}|^2}{2} \delta V. \quad (4.30)$$

Using (4.28) and the formula  $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{a}(\vec{b} \cdot \vec{c})$ , the term between brackets is rearranged to

$$\left\{ \left( (\vec{r} \times \vec{s}) \times \vec{h} \right) \times \delta \vec{t} \right\} = \left\{ -(\vec{r} \times \vec{s}) \vec{h} \cdot \delta \vec{t} + \vec{h} (\vec{r} \times \vec{s}) \cdot \delta \vec{t} \right\} \quad (4.31)$$

so that

$$\delta \Phi = -(\vec{r} \times \vec{s})^* \sigma_M \delta \vec{t} \quad , \quad \sigma_M = \mu_0 \left( \vec{h} \vec{h}^* - \frac{|\vec{h}|^2}{2} I \right). \quad (4.32)$$

Note here how different from each other are the representation of a 1-form  $\vec{h}$  (4.29) and the representation of a 2-form  $\vec{b}$  (4.22). Consequently, their behaviour under the perturbation  $\delta \vec{t}$  is quite different as well, i.e.

$$\delta \vec{b} = \frac{1}{V} \left( -\vec{b} (\vec{r} \times \vec{s}) \cdot \delta \vec{t} + \vec{b} \cdot (\vec{r} \times \vec{s}) \delta \vec{t} \right) \quad , \quad \delta \vec{h} = \frac{1}{V} \left( -(\vec{r} \times \vec{s}) \vec{h} \cdot \delta \vec{t} \right) \quad (4.33)$$

although they are considered to be of an exactly identical nature in the context of vector electromagnetism. This has important implications on the electro-mechanical behaviour of magnetic materials as will be shown in section 7.

## 5 Virtual work principle

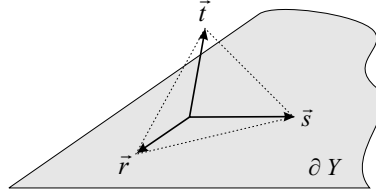


Figure 2: Tetrahedral box.

The local EM forces formulae stemming from the application of the virtual work principle are straightforwardly found back by applying the same procedure to the *tetrahedral* box defined by the same vectors  $\vec{r}$ ,  $\vec{s}$  and  $\vec{t}$ , Fig. 2. The only differences are a few scalar factors arising from the fact that the facets are halved and the volume of the tetrahedron is  $V' = V/6$ . Inside the tetrahedron, the coordinate  $\gamma$  (2.11) is a linear function of  $\vec{x}$  that is zero at all vertices excepted the tip of  $\vec{t}$ . It is therefore a barycentric first order nodal shape function for the deforming tetrahedron. Equation (4.26) can be rewritten

$$\delta \Psi = V' \vec{f}^* \delta \vec{t}, \quad \vec{f} = (\vec{\nabla}^* \gamma) \sigma_M, \quad f_j = \frac{1}{\mu_0} \frac{\partial \gamma}{\partial x_i} \left( b_i b_j - \frac{|\vec{b}|^2}{2} \delta_{ij} \right), \quad (5.34)$$

which is equivalent to the expression found in [14].

## 6 Applicability conditions

The calculations of the last two sections being concise and self-supporting, the applicability conditions show up more clearly and can therefore be reviewed.

The description of the electro-mechanical coupling is completely contained in the chosen expression, at each point, of the (co)energy density of the material. Consequently, the expression of the local *EM force density* as the divergence of the calculated Maxwell stress tensor, i.e.  $\vec{f}_M = (\vec{\nabla}^* \sigma_M)^*$  is, according to

this approach, an outcome rather than a postulate. Another consequence is that each material has its own expression of the Maxwell stress tensor.

The right level to tackle with the definition of EM forces is the infinitesimal, but finite, box we have chosen. At the limit, the *Maxwell stress tensor* is then found to be the truly local expression of EM forces. At the continuous level, it can be directly used as an applied stress in a structural analysis and it can be integrated over a surface. At the discrete level however, it is advisable to prefer implementations that involve a layer of finite elements rather than those which rely on the values on a surface only of the fields. In contrast, the application of the virtual work principle (5.34) is not local. It involves explicitly the coordinate  $\gamma$ , which depend on the particular choice of the box. In practice, the virtual work principle is applied at the discrete level, assimilating the box with a (tetrahedral) finite element.

Both techniques admit dual formulations. For variational consistency however, it is required not to mix fields from different formulations when evaluating the EM forces, i.e. for instance, not to mix the  $\vec{h}$  field from a  $\vec{h}$ -formulations with the  $\vec{b}$  field from a  $\vec{b}$ -formulations, although this may seem a good idea from the point of view of the individual accuracy of the different fields.

The application of Stokes theorem to the work done by Maxwell stress tensor gives

$$\int_{\partial\Omega} \sigma_M \vec{u} d\partial\Omega = \int_{\Omega} \vec{\nabla}^*(\sigma_M \vec{u}) d\Omega = \int_{\Omega} \left( \vec{f}_M \cdot \vec{u} + \sigma_M \cdot (\vec{\nabla} \vec{u}_X^*) \right) d\Omega. \quad (6.35)$$

The force obtained by integrating Maxwell stress tensor  $\sigma_M$  over a closed surface placed in the air and surrounding a moving piece is therefore equal to the resultant EM force  $\vec{f}_M$  only if the piece is perfectly rigid, i.e.  $\vec{\nabla} \vec{u}_X^* = 0$  in  $\Omega$ .

## 7 Magnetic materials

It is not the purpose of this paper to give a detailed analysis of the electro-mechanical coupling in specific magnetic materials. There is room however to show, with two examples, how the proposed approach helps answering the question : How is the magnetic law affected by deformation ?

The constitutive law of *permanent magnet material* and the corresponding EM co-energy density are

$$\vec{b} = \mu_0(\vec{h} + \vec{m}) \quad , \quad \rho^\Phi(\vec{h}) = \mu_0 \left( \frac{\vec{h} \cdot \vec{h}}{2} + \vec{m} \cdot \vec{h} \right) \quad (7.36)$$

where the magnetisation vector  $\vec{m}$  is constant. What is now the kind of vector of  $\vec{m}$ ? According to whether one decides to have  $\vec{m}$  of the same kind as  $\vec{h}$  or  $\vec{b}$ , the Maxwell stress tensor in the magnet is respectively, by application of the procedure described in section 4,

$$\sigma_M = \mu_0 \left( \left( \frac{\vec{h} \cdot \vec{h}}{2} + \vec{h} \cdot \vec{m} \right) I - (\vec{h} \vec{h}^* + \vec{m} \vec{h}^* + \vec{h} \vec{m}^*) \right), \quad \sigma_M = \mu_0 \left( \left( \frac{\vec{h} \cdot \vec{h}}{2} - \vec{h} \cdot \vec{m} \right) I - \vec{h} \vec{h}^* \right). \quad (7.37)$$

This shows that the distinction between 1-forms and 2-forms, which is irrelevant in the expression of the *electromagnetic* constitutive laws, becomes essential when the *electro-mechanical* coupling is considered. There is however no mathematical reason to favour one of these expressions. The first one should be better if magnetisation is due to magnetic dipoles whereas the second one should be better if magnetisation is due to flux carriers (such as Abrikosov vortices in HTc Type II superconductors [5]).

As a second example, the following co-energy density functional is considered

$$\rho^\Phi(\varepsilon, \vec{h}) = \int_0^\varepsilon \sigma_{Mech}(\varepsilon) d\varepsilon - \mu_0 \frac{\vec{h} \cdot \vec{h}}{2} - \mu_0 \int_0^{\vec{h}} \vec{m}_{Mag}^* d\vec{h} + \mu_0 \int_0^{\vec{h}} \vec{m}_{Jou}^* \hat{\varepsilon} d\vec{h} \quad (7.38)$$



where  $\vec{m}_{Mag}$  and  $\vec{m}_{Jou}$  are vector-valued functions of  $\vec{h}$  that represent respectively the magnetisation at zero-strain and a kind of *Joule magnetostriction*, as it depends on the deviatoric part of the strain only. The associated *constitutive laws* are

$$\hat{\sigma} = \partial_{\hat{\varepsilon}} \rho^{\phi} = \hat{\sigma}_{Mech} + \mu_0 \int_0^{\vec{h}} \vec{m}_{Jou} d\vec{h}^* \quad , \quad \vec{b} = -\partial_{\vec{h}} \rho^{\Phi} = \mu_0(\vec{h} + \vec{m}_{Mag}) - \mu_0 \hat{\varepsilon} \vec{m}_{Jou}. \quad (7.39)$$

They automatically verify the thermodynamic consistency condition  $\partial_{\hat{\varepsilon}} \vec{b} = -\partial_{\vec{h}} \hat{\sigma}$ , which is generally not the case when the magnetostriction term is *ad hoc* implemented directly from measurements. Applying the same the same principles as in section 4, one can readily derive the Maxwell stress tensor associated with this material. In particular, the variation of the magnetostrictive term gives

$$\delta \left( \mu_0 V \int_0^{\vec{h}} \vec{m}_{Jou}^* \hat{\varepsilon} d\vec{h} \right) = \mu_0 V \vec{m}_{Jou}^* \hat{\varepsilon} \delta \vec{h} + \mu_0 V \int_0^{\vec{h}} \vec{m}_{Jou}^* \delta \hat{\varepsilon} d\vec{h} + \mu_0 \delta V \int_0^{\vec{h}} \vec{m}_{Jou}^* \hat{\varepsilon} d\vec{h} \quad (7.40)$$

with  $\delta \vec{h}$  given by (4.33) and  $\delta \hat{\varepsilon}$  the deviatoric part of  $\vec{\nabla} \vec{u}_X^*$ .

## 8 Comment

In this paper, the description of the local electro-mechanical coupling is written in terms of classical *vector analysis* and *tensor analysis* notions. It is hence restricted to the particular case of an Euclidean 3D space. Nevertheless, it owes a lot to *differential geometry*. In a 3D Euclidean space indeed, the vector dot product stands for the metric, the cross product is linked to the wedge product and the mixed product is related with the volume form. An important borrowing from differential geometry is the fundamental distinction we have made between  $\vec{h}$ -like vector fields (1-forms) and  $\vec{b}$ -like vector fields (2-forms), with their associated integral quantities, i.e. circulation and flux respectively. The notion of *placement*, as defined in [3, 4], is not the one used in this approach. Following [8], it is actually decomposed into the manifold map  $\varphi$ , which ensures continuity of matter, and the displacement field  $\vec{u}$ , which is analysed by means of *group theory*.

## 9 Conclusion

This paper introduces the algebraic tools that are needed to analyse the electro-mechanical coupling in the context of a finite element formulation. The needed concepts that stem from differential geometry have been reformulated in terms of classical notions in the particular case of a 3D Euclidean space. Electromagnetic force formulae are deduced directly from their fundamental thermodynamic origin. The link between the Maxwell stress tensor and the virtual work principle is discussed in detail. The meaning of the expression “derivation with fluxes (or currents) held constant” which is used when applying the virtual work principle has been clarified and given its theoretical justification.

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