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An Algebraic Multilevel Preconditioner for Field-Circuit Coupled Problems

D. Lahaye, S. Vandewalle and K. Hameyer

Abstract— **Quasi stationary magnetic field formulations are often coupled with lumped parameter models for the driving electrical system. The finite element discretization of such formulations yields linear systems with a large sparse coefficient matrix bordered by dense coupling blocks. The presence of these blocks prevents the straightforward application of black box algebraic multigrid solvers. We present a modified multigrid cycle that takes the coupling blocks into account. The resulting algebraic multigrid solver is used as a preconditioner for the conjugate gradient method for complex symmetric systems. We give evidence of the efficiency of the new method for the calculation of an induction motor.**

Index Terms—**eddy currents, finite element methods, iterative methods**

I. Introduction

Hybrid field-circuit coupled problems frequently arise in electromagnetic engineering applications. In such problems a differential equation problem for the magnetic field is coupled with a linear system modeling the electrical excitations. We consider two dimensional quasi stationary magnetic field problems and introduce the phasor of the z-component of the magnetic vector potential as field unknown. The electrical circuit on the other hand is formulated in terms of linearly independent Kirchoff current and voltage laws and gives relations between the currents and voltages in the electrically conducting parts of the model. The field and circuit formulations are coupled by the magnetically induced currents and voltages in the conductors. The finite element discretization of the coupled problem results in large system of algebraic equations. Solving such systems linear system forms a computational bottleneck in finite element models for technically relevant problems. Our aim is to alleviate this bottleneck by using efficient algebraic multigrid techniques.

II. Field-Circuit Coupled Problems

To formulate the field-circuit coupled problem, we consider a two dimensional domain Ω partitioned into electrically conducting and non-conducting regions. This domain represents for instance the cross-section of an electrical machine or transformer. The conducting region of Ω is the union of the cross-sections of stranded and solid conductors $\Omega_{str,\,p}$ and $\Omega_{sol,\,q}$. Denoting the non-conduction region by Ω_{core}, we have

$$\Omega = (\cup_p \Omega_{str,\,p}) \cup (\cup_q \Omega_{sol,\,q}) \cup \Omega_{core}\,. \tag{1}$$

We assume all electromagnetic quantities to vary sinusoidally in time at low angular frequency ω. This allows us to write a generic electromagnetic quantity $F(\mathbf{x},t)$ as

$$F(\mathbf{x},t) = \widehat{F}(\mathbf{x})\, e^{\,j\,\omega\,t}\,, \tag{2}$$

where $\widehat{F}$ is the phasor of F. With each stranded and solid conductor we associate the current and voltage drop $\triangle \hat{I}_{str,p}$ and $\triangle \hat{V}_{sol,q}$ respectively. We denote the number of windings and the area of the stranded conductor by $N_{t,p}$ and $S_{str,p}$ and the length of all solid conductors by ℓ_z respectively. Finally, we denote by ν and σ the magnetic reluctivity and the electric conductivity. The magnetic field problem is formulated using the z-component of the magnetic vector potential $\hat{A}_z$. By introducing the notation

$$\mathcal{L}(\hat{A}_z) = -\frac{\partial}{\partial x}\left(\nu\,\frac{\partial \hat{A}_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\nu\,\frac{\partial \hat{A}_z}{\partial y}\right)\,, \tag{3}$$

the magnetic field problem on Ω can be stated as

$$\begin{aligned} \mathcal{L}(\hat{A}_z) + j\,\omega\,\sigma\,\hat{A}_z &= \frac{\sigma}{\ell_z}\;\triangle \hat{V}_{sol,\,q} && \text{on } \Omega_{sol,\,q} \\ \mathcal{L}(\hat{A}_z) &= \frac{N_{t,\,p}}{S_{str,\,p}}\;\triangle \hat{I}_{str,\,p} && \text{on } \Omega_{str,\,p} \\ \mathcal{L}(\hat{A}_z) &= 0 && \text{on } \Omega_{core}\,, \end{aligned} \tag{4}$$

supplied with appropriate boundary conditions [1]. This differential problem allows to model saturation by using techniques explained in e.g. [2]. It is discretized by first order nodal finite elements defined on adaptively constructed meshes of triangles with characteristic mesh width h. The discretization yields the linear system of algebraic equations

$$\mathbf{A}_h\,\mathbf{x}_h = \mathbf{f}_h\,, \tag{5}$$

where $\mathbf{A}_h$ represents the discretized differential operator and where $\mathbf{f}_h$ and $\mathbf{x}_h$ correspond to the electrical excitations and the discrete vector potential respectively. The matrix $\mathbf{A}_h$ is a sparse complex symmetric matrix with spectrum lying in the first quadrant of the complex plane. For a given right-hand side vector $\mathbf{f}_h$, the system (5) can be solved for $\mathbf{x}_h$. In general however, the vector $\mathbf{f}_h$ is unknown a priori as it is a function of way the conducting parts in Ω are interconnected and connected to other circuit elements. It is therefore necessary to take the electrical circuit into account.

A topological method that allows to obtain a description of the circuit in terms of a maximum set of linearly independent Kirchoff current and voltage laws is described in [3]. It operates on a graph T associated with the circuit and allows to treat

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arbitrary interconnections of solid and stranded conductors. For stationary currents, it yields the linear system

$$\mathbf{C}\,\mathbf{y} = \mathbf{g}\,, \tag{6}$$

where the matrix $\mathbf{C}$ represents the fundamental loop and cutset equations associated with the tree T [4], and where $\mathbf{g}$ and $\mathbf{y}$ correspond to the known and unknown voltages and currents in the circuit respectively. The matrix $\mathbf{C}$ is complex symmetric.

The electrical circuit system (6) needs to be generalized to situations where branches are magnetically coupled by the finite element model. The magnetically induced effects in conductors must then be taken into account. This is done by adding the term $(\mathbf{B}_h)^T\,\mathbf{x}_h$ corresponding to the integral of the discrete magnetic vector potential over the cross-section of the solid and stranded conductor, to the left-hand side of (6) to obtain

$$(\mathbf{B}_h)^T\,\mathbf{x}_h + \mathbf{C}\,\mathbf{y} = \mathbf{g}\,. \tag{7}$$

The rectangular matrix $\mathbf{B}_h$ allows to rewrite the magnetic source term $\mathbf{f}_h$ in (5) in terms of the circuit unknowns as

$$\mathbf{f}_h = \mathbf{B}_h\,\mathbf{y}\,. \tag{8}$$

The two linear systems (5) and (7) can therefore be written as the single linear system

$$\mathcal{A}_h \begin{pmatrix} \mathbf{x}_h \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{g} \end{pmatrix}\,, \tag{9}$$

where the matrix $\mathcal{A}_h$ is complex symmetric and has the following block structure

$$\mathcal{A}_h = \begin{pmatrix} \mathbf{A}_h & \mathbf{B}_h \\ (\mathbf{B}_h)^T & \mathbf{C} \end{pmatrix}\,. \tag{10}$$

The field-circuit coupling is thus performed without generating any fill in the discrete differential operator $\mathbf{A}_h$. The dimension of $\mathbf{C}$ (up to a few hundred) is much smaller than that of $\mathbf{A}_h$ (up to one million). The matrix $\mathbf{C}$ furthermore remains unchanged as the mesh is adaptively refined.

This paper deals with solving the linear system (9) efficiently.

III. Algebraic Multigrid

In this section we briefly summarize the multigrid idea while paying particular attention to algebraic multigrid methods. For more detailed information we refer to the text books [5], [6]. Multigrid methods are efficient iterative techniques for solving discretized partial differential equations. They complement the action of a smoother on a given fine grid with the computation of a correction on a coarser grid. The implementation of the required coarser grid discretizations is cumbersome in realistic engineering applications. Algebraic multigrid (AMG) solvers [7] cure this problem by providing algorithms for the automatic (i.e. without user intervention) construction of the coarser grid problem. As algebraic multigrid codes require no information on the geometry of the model, it is easy to incorporate them into existing finite element simulation packages.

To describe algebraic multigrid formally, let Ω^h denote the set of nodes of a computational grid with typical measure of the mesh width h. The discretization of a scalar elliptic PDE on this mesh results in a linear algebraic system $\mathbf{A}_h\,\mathbf{x}_h = \mathbf{b}_h$. In solving this system using AMG solvers, one distinguishes two phases. In a setup phase, the algorithm constructs a hierarchy of coarser meshes and the corresponding linear systems. In the cycling phase, this hierarchy of discrete problems is used to solve the problem by multigrid cycling.

The setup phase consists of the following three steps. First a set of coarse grid points is selected. Then the restriction and interpolation operators mapping from fine to coarse grid are constructed. Finally, a coarse grid equivalent of the fine grid system matrix is constructed.

The selection of coarse grid points induces a partitioning $\Omega^h = C^h \cup F^h$, where C^h and F^h denote the fine and coarse grid points respectively. The construction of this partitioning is referred to as the C/F splitting of Ω^h. The next coarser grid Ω^H is identified with C^h.

After having constructed the C/F splitting, AMG computes a matrix dependent interpolation I_H^h. For both the coarsening and the computation of the interpolation weights, AMG exploits information about the strength of coupling between the nodes in the computational grid. This information is coded in the system matrix $\mathbf{A}_h$. For symmetric problems the restriction operator I_h^H is defined as the transpose of the interpolation, i.e. $I_h^H = (I_H^h)^T$. Having the intergrid transfer operators available, the coarse grid equivalent of $\mathbf{A}_h$ is computed by a Galerkin product

$$\mathbf{A}_H = I_h^H\,\mathbf{A}_h\,I_H^h\,. \tag{11}$$

The above procedure is applied recursively using as input $\mathbf{A}_H$ to construct the next coarser grid problem. The recursion terminates if either the size of $\mathbf{A}_H$ on some coarser level drops below a prescribed number or if the fill-in produced by (11) in $\mathbf{A}_H$ becomes too large.

Algebraic multigrid solvers were originally developed to solve symmetric positive definite problems [8]. In [9], we extended the applicability of AMG for solving two dimensional quasi stationary eddy current magnetic field problems. These problems yield linear systems with complex symmetric coefficient matrices. To solve such problems by AMG, we base the selection of the coarser grid and the computation of the interpolation operator on the real part of the matrix. This interpolation is real, and as a consequence, the coarse grid operator $\mathbf{A}^H$ is again complex symmetric. Once the coarse grid problem is constructed, multigrid cycling in complex arithmetic can be performed.

The straightforward application of AMG to system (9) involving the matrix $\mathcal{A}_h$ is hampered by the presence of the submatrices $\mathbf{B}_h$ and $\mathbf{C}$. These submatrices destroy the structure of the real part of the system matrix for which AMG is known to perform satisfactorily. In developing performant iterative schemes for solving (9), we first tried to reuse an existing AMG code without altering its black-box nature. We did so by incorporating the AMG code in block Jacobi and Gauss-Seidel preconditioning schemes. In an attempt to improve this approach, we generalized the AMG scheme. In this generalized scheme the sequence of coarser grids is built based on the real part of the submatrix $\mathbf{A}_h$ and the matrices $\mathbf{B}_h$ and $\mathbf{C}$ are taken into account on the coarsest grid and in the cycling phase. The block

preconditioning and generalized AMG approaches will be discussed in Sections IV and V respectively.

In all our numerical experiments we make use of the AMG code developed by K. Stüben [7]. For the implementation of the algorithms we will present in this paper, we wrote an interface that allows to call this AMG code from within PETSC [10].

IV. Block Preconditioning Schemes

A first approach in reusing AMG for solving (9) is through the use of *block* preconditioning techniques. The block Jacobi scheme

$$\begin{pmatrix} \mathbf{A}_h & 0 \\ 0 & \mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix} \tag{12}$$

can be used as a preconditioner for the CG algorithm for complex symmetric systems [11]. The matrix $\mathbf{B}_h$ can be taken into account by switching to the non-symmetric Gauss-Seidel preconditioner

$$\begin{pmatrix} \mathbf{A}_h & 0 \\ (\mathbf{B}_h)^T & \mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{z}_1 \\ \mathbf{z}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \end{pmatrix} . \tag{13}$$

for Krylov subspace methods for non symmetric problems such as GMRES or BiCGSTAB [12]. The Gauss-Seidel preconditioning step is more expensive than that of Jacobi by multiplication of the matrix $\mathbf{B}_h$ only. The application of the block Jacobi and Gauss-Seidel schemes requires solving a linear system with coefficient matrices $\mathbf{A}_h$ and $\mathbf{C}$ at every step of the Krylov subspace method. As the size of $\mathbf{C}$ is much smaller than that of $\mathbf{A}_h$, the cost of solving the $\mathbf{C}$ linear system is negligible compared to solving the $\mathbf{A}_h$ linear system. In solving the $\mathbf{A}_h$ system, the AMG algorithm for complex symmetric systems developed in [9] can be reused . The system can be solved either exactly or approximately by applying a few (but fixed) number of cycles. Another alternative consists in making the accuracy of the inner AMG solve function of the residual norm of the outer Krylov iteration. The resulting variable preconditioner can be accelerated by Flexible GMRES (FGMRES) [12] for instance.

For the implementation of the above algorithms, we made use of the block preconditioning framework and Krylov subspace solvers available in PETSC. The interface between AMG and PETSC allows to (approximately) solve the $\mathbf{A}_h$ within each step of the block-precondioner by AMG.

V. Algebraic Multigrid for Field-Circuit Coupled Problems

Our multigrid technique for solving field-circuit coupled problems is a generalization of the method for solving an elliptic problem augmented by an algebraic equation found in [5, Section 11.4].

Let the linear system (9) with coefficient matrix $\mathcal{A}_h$ defined in (10) be a given fine grid discretization of the coupled problem. Let Ω^h denote the set of fine grid degrees of freedom. The set Ω^h embraces both the magnetic and electric variables. Denoting the former and the latter by Ω^h_M and Ω^h_E respectively, we have that

$$\Omega^h = \Omega^h_M \cup \Omega^h_E . \tag{14}$$

Each variable in Ω^h_M corresponds to a finite element mesh point. The number of elements of Ω^h_E equals the number of loop and cutset equations. Next we describe a generalised AMG algorithm for solving the the given linear system. We will first give details of the two-grid variant of our algorithm. In doing so, we discuss the setup and the solve phase separately.

We coarsen the set Ω^h is such a way that the electric degrees of freedom Ω^h_E are in the coarse grid, i.e.,

$$\Omega^h_E \subset \Omega^H . \tag{15}$$

The magnetic degrees of freedom Ω^h_M are split into coarse and fine ones denoted by C^h_M and F^h_M respectively. The magnetic coarse grid Ω^H_M is identified with C^h_M. AMG constructs this splitting and the magnetic interpolation I^h_H mapping from Ω^H_M to Ω^h_M using information contained in the real part of the first diagonal block of $\mathcal{A}_h$, i.e. in the real part of the discrete differential operator $\mathbf{A}_h$. The next coarser grid Ω^H and the interpolation operator $\mathcal{I}^h_H$ for the coupled problem can now be introduced. The set Ω^H is defined as follows

$$\Omega^H = \Omega^H_M \cup \Omega^h_E . \tag{16}$$

This definition implies that the interpolation $\mathcal{I}^h_H$ has the following block diagonal form

$$\mathcal{I}^h_H = \begin{pmatrix} I^h_H & 0 \\ 0 & I \end{pmatrix} , \tag{17}$$

where the second diagonal block denotes the identity on Ω^h_E. The symmetry of $\mathcal{A}_h$ motivates one to define the restriction $\mathcal{I}^H_h$ as transpose of the interpolation. The coarse grid equivalent of $\mathcal{A}_h$ is computed by a Galerkin product, resulting in

$$\mathcal{A}_H = \mathcal{I}^H_h \, \mathcal{A}_h \, \mathcal{I}^h_H = \begin{pmatrix} \mathbf{A}_H & \mathbf{B}_H \\ (\mathbf{B}_H)^T & \mathbf{C} \end{pmatrix} , \tag{18}$$

where $\mathbf{A}_H = I^H_h \, \mathbf{A}_h \, I^h_H$ and $\mathbf{B}_H = I^H_h \, \mathbf{B}_h$.

In the solve phase, the hierarchy of coarser discretizations constructed in the setup phase is used to solve the given linear system by multigrid cycling. In this phase, we perform smoothing on the magnetic variables only and leave the electric variables unchanged. Given a right-hand side vector $(\mathbf{f}_h, \mathbf{g})$ and a start solution $(\mathbf{x}^0_h, \mathbf{y}^0)$ for the linear system (9), smoothing consists of computing a modified magnetic right-hand side term $\overline{\mathbf{f}}_h = \mathbf{f}_h - \mathbf{B}_h \, \mathbf{y}^0$ and applying Gauss-Seidel smoothing to the system $\mathbf{A}_h \, \mathbf{x}_h = \overline{\mathbf{f}}_h$. The coarse grid correction is computed by solving the linear system with matrix (18) by a direct solver.

If the two-grid scheme is applied recursively to solve this coarse grid system, a multi-grid scheme is obtained. In our numerical experiments we use this multigrid scheme as a preconditioner for the conjugate gradient algorithm for complex symmetric systems [11].

For the implementation of the above algorithm, we developed the interface between AMG and PETSC already mentioned in Section IV. In the resulting code only the setup part of Stüben's AMG code is used. To perform the multigrid cycling, the multigrid framework within PETSC was extended to accommodate the circuit relations.

VI. A Practical Example

To test the efficiency of the algorithm, a model of a 45kW induction machine is taken as example. The equipotential lines of the real part of the computed magnetic vector potential are shown in Fig.1. The final mesh was obtained after three adaptive refinement steps and contains a total of 118802 elements and 59574 nodes. The electrical circuit is modeled by 148 equations.

For this example we compare the performance of the block Jacobi and the generalized AMG scheme. Both methods are accelerated by the conjugate gradient method. In the block Jacobi preconditioner, we approximately solve the $\mathbf{A}_h$ linear system by applying just one V multigrid cycle with one pre and one post smoothing step. The same cycle is used in the generalized AMG scheme. Hence, both algorithms have about the same computational complexity per iteration step. In Fig. 2 we plotted the number of iterations required by both schemes to reduce the residual by a factor of 10^{-12}. This figure shows that the number of iterations required by both schemes is mesh-width independent. It also shows that the generalized AMG scheme is superior to the block Jacobi scheme in terms of number of iterations by a factor of about 2.

In Fig. 3 we compare the CPU time required by the generalized AMG scheme with a standard ILU preconditioner taken from PETSC. This figure shows that the use of the generalized AMG scheme yields a considerable speedup that becomes more significant with increasing mesh size. On the finest grid considered in this numerical test, AMG speeds up calculations by a factor of almost 4.

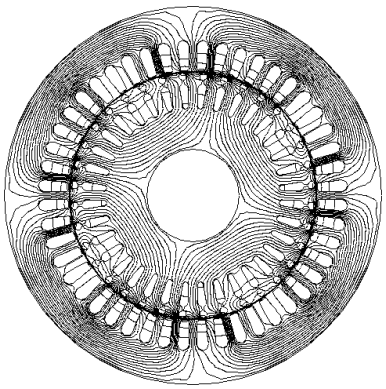

Fig. 1. Equipotential lines of the real part of the computed vector potential.

VII. Conclusions

We presented an algebraic multigrid preconditioner for time harmonic field-circuit coupled problems. In the calculation of an induction machine, the use of the multigrid preconditioner resulted in a significant acceleration compared to an ILU preconditioned conjugate gradient solver.

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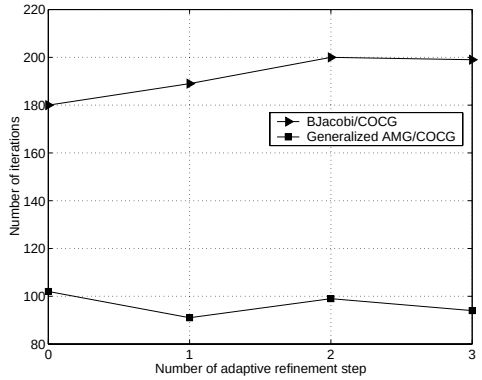

Fig. 2. Number of iterations of the block Jacobi and generalized algebraic multigrid schemes versus the number of adaptive refinement step.

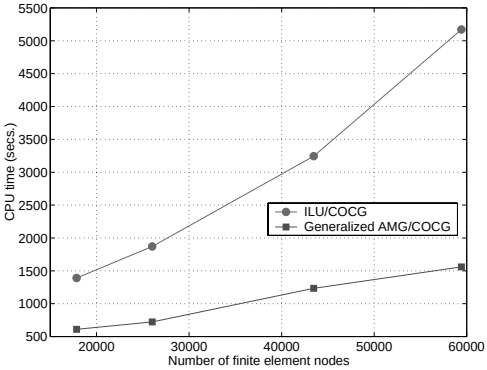

Fig. 3. CPU time of generalized algebraic multigrid and ILU preconditioned conjugate gradient method for complex symmetric systems versus the number of finite element grid points.

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