Newton and quasi-newton algorithms for non-linear electromagnetic-thermal coupled problems

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***Abstract* — A general framework for the application of the Newton methods in non-linear coupled electromagnetic-thermal problems solved with the FEM on independent subproblem meshes is presented. The explicit derivation of the Jacobian matrix is outlined and discussed. A matrix-free quasi-Newton method, to be used along with linear system solvers built around Jacobian-vector products is presented. This method does not require explicit derivatives and can be parallelised. The numerical aspects of these methods are discussed. The different Newton methods are demonstrated using a steady-state conductive heating example problem.**

I. Introduction

Non-linear coupled field problems, more in particular thermal-magnetic fields are usually solved by means of the finite element method using Picard algorithms (successive substitution) [1]. This calculation methodology consists of iterating over a sequence of more or less standard subproblem solutions. The intermediate operations contain loss calculation algorithms and the calculation of the thermal influence upon the local material characteristic, occurring as a dependent coefficient. Different subproblem meshes can be combined using projections in between the calculations.

This paper tries to give a general framework to compose the Jacobian required for the non-linear Newton iterations for a combined frequency domain magnetic-thermal problem on independent subproblem meshes. A quasi-Newton approach will be presented.

In the derivation, the coupled thermal-magnetic problem will be represented as in (1).

$$\begin{cases} G_A(A, T') = 0 \\ G_T(A', T) = 0 \end{cases} \tag{1}$$

G_A denotes the magnetic field residual, written in terms of the magnetic vector potential and G_T the thermal field equation residual. In case of a magnetic frequency domain method, A is a complex quantity. The coupling functions, the dependent material characteristics and loss functions are implicitly included in (1). Here the thermal dependent electrical conductivity and joule losses are considered. The variables A' and T' stand for the partial solution (implicitly) projected on the mesh of the corresponding sub-problem.

II. Full Newton Approach

A. Derivation

In the Newton-Raphson method, a set of corrections δ_A and δ_T to be added to an estimate of the non-linear solution is calculated [2]-[4]. The system to be solved contains the Jacobian matrix and the residuals evaluated in the estimated solution.

$$\begin{bmatrix} \dfrac{\partial G_A}{\partial A} & \dfrac{\partial G_A}{\partial T'} \cdot \dfrac{\partial T'}{\partial T} \\ \dfrac{\partial G_T}{\partial A'} \cdot \dfrac{\partial A'}{\partial A} & \dfrac{\partial G_T}{\partial T} \end{bmatrix} \cdot \begin{bmatrix} \delta_A \\ \delta_T \end{bmatrix} = -\begin{bmatrix} G_A \\ G_T \end{bmatrix} \tag{3}$$

However, when a frequency domain method is used, G_A is complex and not analytic [5]:

$$\frac{\partial \operatorname{Re}\{G_A\}}{\partial \operatorname{Re}\{A\}} \neq \frac{\partial \operatorname{Im}\{G_A\}}{\partial \operatorname{Im}\{A\}}, \quad \frac{\partial \operatorname{Re}\{G_A\}}{\partial \operatorname{Im}\{A\}} \neq -\frac{\partial \operatorname{Im}\{G_A\}}{\partial \operatorname{Re}\{A\}} \tag{4}$$

As a consequence, the derivatives cannot be determined directly. A solution to this is to split up the residual and treat the real and imaginary components independently:

$$\begin{bmatrix} \dfrac{\partial \operatorname{Re}\{G_A\}}{\partial \operatorname{Re}\{A\}} & \dfrac{\partial \operatorname{Re}\{G_A\}}{\partial \operatorname{Im}\{A\}} & \dfrac{\partial \operatorname{Re}\{G_A\}}{\partial T'} \cdot \dfrac{\partial T'}{\partial T} \\ \dfrac{\partial \operatorname{Im}\{G_A\}}{\partial \operatorname{Re}\{A\}} & \dfrac{\partial \operatorname{Im}\{G_A\}}{\partial \operatorname{Im}\{A\}} & \dfrac{\partial \operatorname{Im}\{G_A\}}{\partial T'} \cdot \dfrac{\partial T'}{\partial T} \\ \dfrac{\partial G_T}{\partial \operatorname{Re}\{A'\}} \cdot \dfrac{\partial \operatorname{Re}\{A'\}}{\partial \operatorname{Re}\{A\}} & \dfrac{\partial G_T}{\partial \operatorname{Im}\{A'\}} \cdot \dfrac{\partial \operatorname{Im}\{A'\}}{\partial \operatorname{Im}\{A\}} & \dfrac{\partial G_T}{\partial T} \end{bmatrix} \cdot \begin{bmatrix} \operatorname{Re}\{\delta_A\} \\ \operatorname{Im}\{\delta_A\} \\ \delta_T \end{bmatrix} = -\begin{bmatrix} \operatorname{Re}\{G_A\} \\ \operatorname{Im}\{G_A\} \\ G_T \end{bmatrix} \tag{5}$$

with the same projection derivatives (geometrical functions):

$$\frac{\partial A'}{\partial A} = \frac{\partial \operatorname{Re}\{A'\}}{\partial \operatorname{Re}\{A\}} = \frac{\partial \operatorname{Im}\{A'\}}{\partial \operatorname{Im}\{A\}} \tag{6}$$

These derivatives may introduce many entries in the Jacobian as the projection operation relates to many degrees of freedom on the different meshes. For instance, the degree of freedom associated to the node x_0 in Fig. 1, will in general depend on nodes x_1', x_0' and x_5'. This results in a higher off-diagonal fill-in.

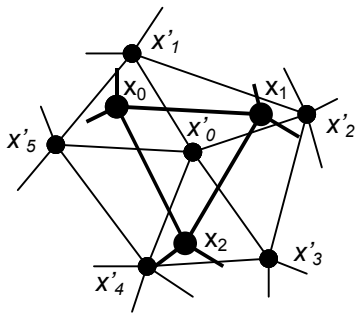


Fig.1. Mesh nodes coupled by projection, occurring on the different subproblem meshes.

In many cases, the explicit determination of the partial derivatives is a major problem as often the losses are computed through a complicated procedure and the material data are represented by tables.

The block-structured Jacobians in (3) and (5) are asymmetrical. Realistic problems are often ill-conditioned. This can be identified by the high condition number. Therefore, advanced iterative linear system solvers, such as GMRES [6] are required.

II. Quasi-Newton Approaches

A. Approximation methods

The computational cost of a Newton method is determined by the effort required for the construction of the Jacobian and the solution of the correction equation system. As the full Newton method can become a very expensive method, many approximations or quasi-Newton methods have been proposed [3][7], for instance the approximation of the Jacobian matrix.

An interesting approach is to approximate the product of the Jacobian and a (small) vector. This type of matrix-vector product occurs in iterative linear system solvers such as GMRES or TFQMR [6]. A way to accomplish this is to use the difference of the operating point (partial) residual and a perturbed (partial) residual:

$$J(x_j)\cdot v \approx \frac{G_i(x_j+\kappa v)-G_i(x_j)}{\kappa} \tag{7}$$

In practice, only the perturbed residual is to be calculated, which requires the solution of the considered problem, since the operating point residual is initially computed for the right-hand side of the correction equation system. Hence, a Jacobian matrix-free algorithm or implicit Jacobian method arise. This approach was initially suggested for fluid dynamics calculations [8], but is suggested in other scientific domains as well.

Obviously, the accuracy of the approximation depends on the choice of κ. It influences the approximation as well as the round-off errors. A good choice, using the machine precision ε is [7] [8]:

$$\kappa = 2\varepsilon^{1/2}\max\left(|x_j|, magn(x_j)\right)\|v\|^{-1/2} \tag{8}$$

B. Implementation of Jacobian matrix-free algorithm

However, to use the difference approximation of (7) along with the Jacobian in (5), several evaluations of a partial residual, at least one for every subproblem block, are required. Moreover, the use of the GMRES algorithm will lead to a large memory consumption.

Therefore, it is often more interesting to rewrite the equations explicitly in terms of a smaller set of variables, the 'coupling variables'. Here, the set of electrical conductivities and loss quantities connected to the finite element mesh is appropriate:

$$\begin{cases} q - G_q(\sigma) = 0 \\ \sigma - G_\sigma(q) = 0 \end{cases} \tag{9}$$

Then, the correction equation becomes:

$$\begin{bmatrix} I & -\frac{\partial G_q}{\partial \sigma} \\ -\frac{\partial G_\sigma}{\partial q} & I \end{bmatrix} \cdot \begin{bmatrix} \delta_q \\ \delta_\sigma \end{bmatrix} = -\begin{bmatrix} q^* - G_q(\sigma^*) \\ \sigma^* - G_\sigma(q^*) \end{bmatrix} \tag{10}$$

Using the difference approximation (7), the matrix-vector product in GMRES reduces to:

$$\begin{bmatrix} I & -\frac{\partial G_q}{\partial \sigma} \\ -\frac{\partial G_\sigma}{\partial q} & I \end{bmatrix} \cdot \begin{bmatrix} v_q \\ v_\sigma \end{bmatrix} \approx \begin{bmatrix} v_q + \frac{G_q(\sigma^*) - G_q(\sigma^* + \kappa_\sigma v_\sigma)}{\kappa_\sigma} \\ v_\sigma + \frac{G_\sigma(q^*) - G_\sigma(q^* + \kappa_q v_q)}{\kappa_q} \end{bmatrix} \tag{11}$$

The calculation of the perturbed partial residuals can be performed in parallel, yielding an acceleration of about 40 %. The operation point solution can be used as a starting solution for this calculation.

II. Application: Conductive Heating

A. Problem description

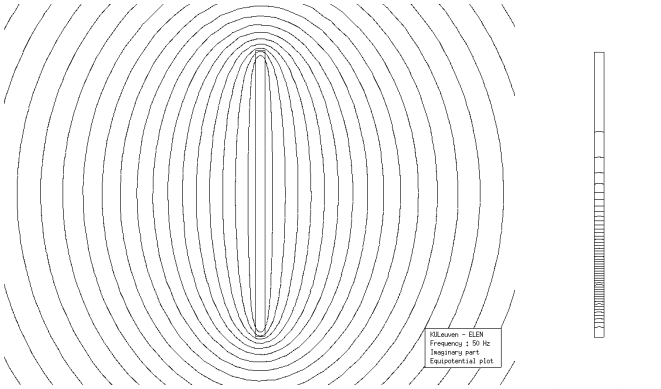
As an application, a free, long, massive busbar with an applied voltage is calculated. The electrical conductivity is temperature dependent. The only losses in the model are caused by the source and eddy currents in the conductor. It is cooled by convection at all faces. One edge is assumed to be cooled with a lower convection coefficient, introducing a moderately asymmetrical cooling. The problem equations are:

$$\nabla\cdot(\upsilon_r \nabla A) - \mu_0 j\omega\sigma(T')A = -\mu_0\sigma(T')V_s \tag{12}$$

$$\nabla\cdot(\lambda\nabla T) = -q_J(T, A') \tag{13}$$

The non-linear coupled problem has (at least) two solutions. The first, physical, solution (Fig. 2) results in a moderate increase of the bar temperature to a hot spot temperature of about 55 °C at the bottom where the highest current density is located. The other solution is impossible in the physical reality, as it is a result of the unconstrained extrapolation of the material characteristics involved in the coupling. Although this solution is not physically viable with this type

of material and cooling conditions, it is not to be excluded that for other materials, configurations and conditions, multiple stable states may exist in the physical reality. In that case, the object's history will determine which state eventually is reached.



magnetic field around the bar — thermal field inside the bar

Fig.2. Magnetic and thermal field solution of the coupled test problem.

Two independent FEM models with first order triangular discretisations are used. Whether the solution converges to the physical or non-physical field, depends on the algorithm, its associated parameters and the starting solution. Fig. 3 illustrates the convergence history of various Picard algorithms [9] with relaxation and the (quasi-) Newton algorithm. The plotted parameter is the relative L_∞-norm of the difference between consecutive solutions. The Newton algorithm converges faster in the vicinity of the exact solution, but is more expensive. The total computation time depends on the software implementation of the different underlying algorithms, but on a non-parallel computer, every step takes a comparable amount of CPU time.

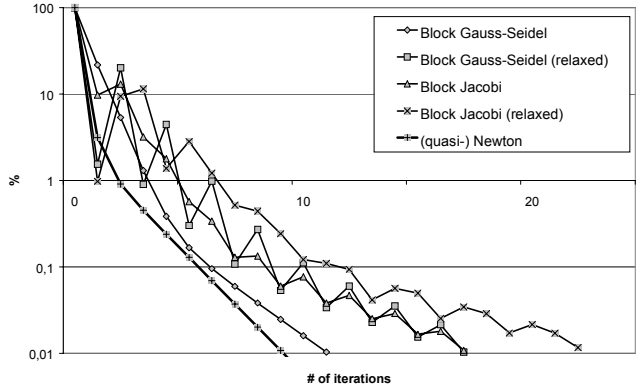


Fig.3. Convergence history of different coupled problem solution algorithms.

B. Jacobian derivation

The magnetic field equation yields a complex FEM matrix system:

$$\left[K_A + jH_A(T')\right]\underline{A} - \underline{F}_A(T') = 0 \tag{14}$$

with K_A and H_A the matrices due to the diffusion and eddy current term in (12). F_A contains the voltage driven source. This equation has to be split in the real and imaginary component:

$$\begin{aligned} K_A A_r - H_A(T')A_i - F_{A_r}(T') &= 0 \\ H_A(T')A_r + K_A A_i - F_{A_i}(T') &= 0 \end{aligned} \tag{15}$$

Likewise, the thermal equation yields:

$$K_T T - F_T(A_r', A_i', T) = 0 \tag{16}$$

with K_T the FEM matrix and F_T the joule loss term.
The Jacobian for the non-linear system consisting of (15) and (16) becomes:

$$\begin{bmatrix} K_A & -H_A & \left(-\frac{\partial H_A}{\partial T'}\cdot\frac{\partial T'}{\partial T}\cdot A_i - \frac{\partial F_{A_r}}{\partial T'}\cdot\frac{\partial T'}{\partial T}\right) \\ H_A & K_A & \left(\frac{\partial H_A}{\partial T'}\cdot\frac{\partial T'}{\partial T}\cdot A_r - \frac{\partial F_{A_i}}{\partial T'}\cdot\frac{\partial T'}{\partial T}\right) \\ -\frac{\partial F_T}{\partial A_r'}\cdot\frac{\partial A_r'}{\partial A_r} & -\frac{\partial F_T}{\partial A_i'}\cdot\frac{\partial A_i'}{\partial A_i} & \left(K_T - \frac{\partial F_T}{\partial T}\right) \end{bmatrix}$$
$$\cdot\begin{bmatrix}\delta_{A_r}\\ \delta_{A_i}\\ \delta_T\end{bmatrix} = -\begin{bmatrix} K_A A_r - H_A A_i - F_{Ar} \\ H_A A_r + K_A A_i - F_{Ai} \\ K_T T - F_T \end{bmatrix} \tag{17}$$

The blocks associated with the eddy current term contain the temperature dependent electrical conductivity. As this temperature is obtained on the thermal mesh, projection is required. These same derivatives show up in the different source terms as well:

$$\frac{\partial H_A}{\partial T'}\cdot\frac{\partial T'}{\partial T} = \frac{\partial H_A}{\partial \sigma}\cdot\frac{\partial \sigma}{\partial T'}\cdot\frac{\partial T'}{\partial T} \tag{18}$$

$$\frac{\partial F_{A_{r,i}}}{\partial T'}\cdot\frac{\partial T'}{\partial T} = \frac{\partial F_{A_{r,i}}}{\partial \sigma}\cdot\frac{\partial \sigma}{\partial T'}\cdot\frac{\partial T'}{\partial T} \tag{19}$$

$$\frac{\partial F_T}{\partial T} = \frac{\partial F_T}{\partial q_J}\cdot\frac{\partial q_J}{\partial \sigma}\cdot\frac{\partial \sigma}{\partial T} \tag{20}$$

The heat density term also depends on the magnetic field, due to the eddy current loss contribution.

$$\frac{\partial F_T}{\partial A_{r,i}'}\cdot\frac{\partial A_{r,i}'}{\partial A_{r,i}} = \frac{\partial F_T}{\partial q_J}\cdot\frac{\partial q_J}{\partial A_{r,i}'}\cdot\frac{\partial A_{r,i}'}{\partial A_{r,i}} \tag{21}$$

It is assumed that the element's material parameters are modelled by constants within the first order element's range. To correct the quantity, the average temperature is used, yielding the following expression for the derivative associated to a certain node:

$$\frac{\partial \sigma}{\partial T_k} = -\frac{\sigma_{ref}\alpha}{\left(1+\alpha\left(T_{eq}-T_{ref}\right)\right)^2}\cdot\frac{\partial T_{eq}}{\partial T_k} = -\frac{\sigma^2}{\sigma_{ref}}\cdot\alpha\cdot\frac{\partial T_{eq}}{\partial T_k} \tag{22}$$

$$\frac{\partial \sigma}{\partial T} = -\frac{\sigma\left(\frac{1}{3}\sum_{j=1}^{3} T_j\right)^2}{\sigma_{ref}}\cdot\frac{\alpha}{3} \tag{23}$$

The joule loss expression is:

$$q_J = \frac{J_{eq} J_{eq}^*}{\sigma} = \sigma \left[\left(V_r + \frac{\omega}{3} \sum_{j=1}^{3} A_{ij} \right)^2 + \left(V_i - \frac{\omega}{3} \sum_{j=1}^{3} A_{rj} \right)^2 \right] \tag{24}$$

yielding the derivatives (Vr and V_i are the real and imaginary parts of V_s):

$$\frac{\partial q_J}{\partial A_{rk}} = \frac{2}{3} \sigma \omega \left(V_i - \frac{\omega}{3} \sum_{j=1}^{3} A_{rj} \right) A_{rk} \tag{25}$$

$$\frac{\partial q_J}{\partial A_{ik}} = \frac{2}{3} \sigma \omega \left(V_r + \frac{\omega}{3} \sum_{j=1}^{3} A_{ij} \right) A_{ik} \tag{26}$$

The Jacobian matrix constructed in this way, has the structure of Fig. 4, in which the different blocks can be identified.

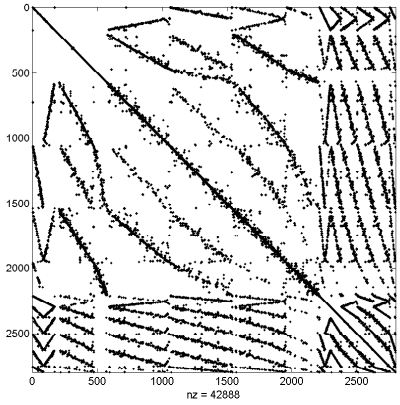


Fig.4. Fill-in pattern of the Jacobian matrix of the benchmark problem.

The condition numbers in this example problem range from 10^8 to 10^{10}, being rather high. The eigenvalue distribution (Fig. 5) is complicated, yielding a slow convergence. The diagonal magnetic and thermal subblocks have condition numbers of about 10^3-10^4.

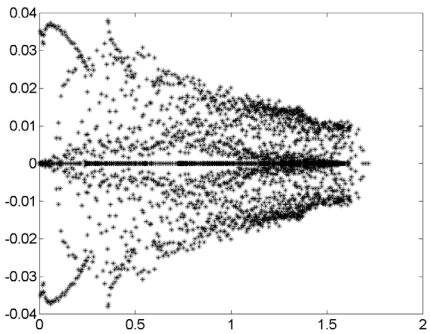


Fig.5. Eigenvalue distribution of the Jacobian matrix in the phase plane.

C. Discussion

The Jacobian-free quasi-Newton implementation, written in terms of the electrical conductivites and loss densities of the elements has a comparable convergence behaviour. In its implementation, it is not required to split the time-harmonic problem into a real and imaginary component as required for the Jacobian. It is also possible to use more complicated material models and loss calculation algorithms in (22) and (24). The problem of the ill-conditioned Jacobian is avoided due to the block arrangement. The acceleration of the algorithm due to the parallelisation seems to compensate for the Jacobian approximation error.

V. CONCLUSION

Full Newton algorithms require the composition of a Jacobian matrix, for which a general methodology is outlined here. This involves the inclusion of field quantities projection equations, joule and eddy-current loss calculation expressions and non-linear thermal dependencies of material characteristics, such as the electrical conductivities. The numerical properties, with respect to the iterative linearised system solution of the resulting correction equation are discussed.

An approximate (quasi-)Newton method, having the advantage that the Jacobian matrix has not to be computed explicitly, is presented. This approach uses a difference approximation for Jacobian-vector products, as occurring in iterative solvers algorithms such as GMRES. A block ordering of the coupled problem equations enables a fast parallel implementation of this approach. An alternative formulation of the coupled problem in terms of element material quantities and loss vectors, yielding a minimal number of non-linear solution variables, is used to limit the memory requirements of this algorithm.

The advantages and drawbacks of these methods are discussed and illustrated using a conductive heating problem.

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