An Algebraic Multilevel Preconditioner for Field-Circuit Coupled Problems

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Abstract—**Time-harmonic magnetic field formulations are often coupled with lumped parameter models of the driving electrical system. The finite element discretization of such formulations yields large linear systems with a sparse matrix bordered by dense coupling blocks. The presence of these blocks prevents the immediate application of fast multigrid solvers. We present a modified multigrid cycle that takes the coupling blocks into account. The resulting algebraic multigrid solver is used as a preconditioner for the conjugate gradient method for complex symmetric systems. We give evidence of the efficiency of the new method in the calculation of an induction motor.**

Keywords—**eddy currents, finite element methods, iterative methods**

I. Introduction

Hybrid field circuit coupled problems frequently arise in electromagnetic engineering applications. We consider two dimensional quasi stationary eddy current problems coupled with a lumped parameter model for the exciting electrical circuit. The partial differential equation governing the phasor of the z-component of the magnetic vector potential $\hat{A}_z$ is the scalar Helmholtz equation with complex shift [1]. This magnetic field equation is discretized by first order triangular nodal finite elements with characteristic mesh width h. The resulting finite element description is tied to a lumped parameter model for the electrical circuit connections. In these electrical relations unknown currents and voltages are associated to solid and stranded conductors respectively [2]. Given the magnetic source term $\mathbf{f}_h$ and the electrical (mesh-width independent) source terms $\mathbf{g}$, the hybrid field-circuit coupled discretization yields the following linear algebraic system for the unknown magnetic and electrical unknowns $\mathbf{x}_h$ and $\mathbf{y}_h$

$$\mathcal{A}_h \begin{pmatrix} \mathbf{x}_h \\ \mathbf{y}_h \end{pmatrix} = \begin{pmatrix} \mathbf{f}_h \\ \mathbf{g} \end{pmatrix} . \qquad (1)$$

The matrix $\mathcal{A}_h$ is complex symmetric and has the following block structure

$$\mathcal{A}_h = \begin{pmatrix} \mathbf{A}_h & \mathbf{B}_h \\ (\mathbf{B}_h)^T & \mathbf{C} \end{pmatrix} \qquad (2)$$

where the submatrices $\mathbf{A}_h$, $\mathbf{B}_h$ and $\mathbf{C}$ represent the finite element discretization of the Helmholtz operator, the field-circuit coupling terms and the electrical circuit respectively. The matrix $\mathbf{C}$ is mesh width independent. The dimension of $\mathbf{C}$ (up to a few hundred) is much smaller than that of $\mathbf{A}_h$ (up to one million).

Solving linear system (1) forms the computational bottleneck in finite element models for technically relevant problems. Our aim is to alleviate this bottleneck by efficient iterative techniques based on the multigrid idea.

II. Algebraic Multigrid

Multigrid methods [3] are efficient iterative techniques for solving discretized partial differential equations. They complement the action of a smoother on a given fine grid with the computation of a correction on a coarser grid. The implementation of the required coarser grid discretizations is cumbersome in realistic engineering applications. Algebraic multigrid (AMG) solvers [4] cure this problem by providing algorithms for the automatic construction of the coarser grid problem.

To describe algebraic multigrid formally, let h and H denote the fine and coarse grid mesh sizes and $\mathbf{A}_h \mathbf{x}_h = \mathbf{b}_h$ the given fine grid discretization of a partial differential equation. Algebraic multigrid partitions the grid nodes into sets of coarse and fine grid nodes and a prolongation operator I_H^h mapping from coarse to fine grid space. The coarse grid operator is built by using the Galerkin formula $\mathbf{A}_H = I_h^H \, \mathbf{A}_h \, I_H^h$, where $I_h^H = (I_H^h)^T$ is the restriction operator. Once the coarse grid problem has been set up, the solution to the fine grid problem can be computed by multigrid cycling. As algebraic multigrid codes require no information on the geometry of the model, it is easy to incorporate them into existing finite element simulation packages.

Algebraic multigrid solvers were originally developed to treat symmetric positive definite problems [5]. In [6] we extended the applicability of AMG for solving two dimensional quasi stationary eddy current magnetic field problems. These problems yield linear systems with complex symmetric coefficient matrices. To solve such problems by AMG, we base the selection of the coarser grid and the computation of the interpolation operator on the real part of the matrix. This interpolation is real, and as a consequence the coarse grid operator $\mathbf{A}^H$ is again complex symmetric. Once the coarse grid problem is constructed, multigrid cycling in complex arithmetic can be performed.

The straightforward application of AMG to the system (1) involving the matrix $\mathcal{A}$ is hampered by the presence of the submatrices B and C. These submatrices destroy the structure of the real part of the system matrix for which AMG is known to perform satisfactorily. We present an AMG approach that builds a sequence of coarser discretizations based on the real part of the

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submatrix A and that takes the matrices B and C into account on the coarsest grid and in the cycling phase.

III. Algebraic Multigrid for Field Circuit Coupled Problems

Our multigrid technique for solving field-circuit coupled problems is a generalization of the method for solving an elliptic problem augmented by an algebraic equation found in [3], Section 11.4.

We describe the two-grid scheme for solving the problem at hand. Let the linear system (1)-(2) be a given fine grid discretization of the coupled problem. In the setup phase the magnetic field equations are coarsened without taking the electrical circuit connections into account. We do not coarsen the electrical circuit matrix. Given the real part of the matrix $\mathbf{A}_h$ as input, the AMG setup algorithm computes a coarse grid for the magnetic unknowns and the corresponding interpolation operator I_H^h. Denoting by I the identity operator for the electric unknowns, we define the interpolation operator for the coupled problem $\mathcal{I}_H^h$ as

$$\mathcal{I}_H^h = \begin{pmatrix} I_H^h & 0 \\ 0 & I \end{pmatrix}, \qquad (3)$$

and the restriction operator $\mathcal{I}_h^H$ as its transpose. The coarse grid equivalent of (2) is then given by

$$\mathcal{A}_H = \mathcal{I}_h^H \, \mathcal{A}_h \, \mathcal{I}_H^h = \begin{pmatrix} \mathbf{A}_H & \mathbf{B}_H \\ (\mathbf{B}_H)^T & \mathbf{C} \end{pmatrix}, \qquad (4)$$

where $\mathbf{A}_H = I_h^H \, \mathbf{A}_h \, I_H^h$ and $\mathbf{B}_H = I_h^H \, \mathbf{B}_h$. In the cycling phase smoothing is performed on the magnetic variables only. Smoothing consists of, given a start solution $(\mathbf{x}_h^0, \mathbf{y}_h^0)$ for the linear system (1), computing a modified magnetic right-hand side term $\overline{\mathbf{f}}_h = \mathbf{f}_h - \mathbf{B}_h \, \mathbf{y}_h^0$ and applying the smoother to the system $\mathbf{A}_h \, \mathbf{x}_h = \overline{\mathbf{f}}_h$. The smoother leaves the electric variables unchanged. The coarse grid correction is computed by solving the linear system with matrix (4) by a direct solver. If the two-grid scheme is applied recursively to solve this coarse grid system, a multi-grid scheme is obtained.

The multigrid is applied as a preconditioner for the conjugate gradient algorithm for complex symmetric systems [7].

For the implementation of the above algorithm we linked K. Stüben's AMG code [4] with PETSc [8] and used PETSc's multigrid components.

IV. A Practical Example

To test the efficiency of our algorithm, a model of a 45kW induction machine is taken as example. The final element mesh was obtained after three adaptive refinement steps and contains a total of 118802 elements and 59574 nodes. The electrical circuit is modeled by 148 equations. The performance of the multigrid preconditioner was compared to an ILU preconditioner taken from PETSc. Figure 1 shows an acceleration in the computation of a factor between 5 and 6.

V. Conclusions

We presented an algebraic multigrid preconditioner for time harmonic field circuit coupled problems. In the calculation of an induction machine, the use of the multigrid preconditioner resulted in an acceleration of a factor between 5 and 6 compared to an ILU preconditioned conjugate gradient solver.

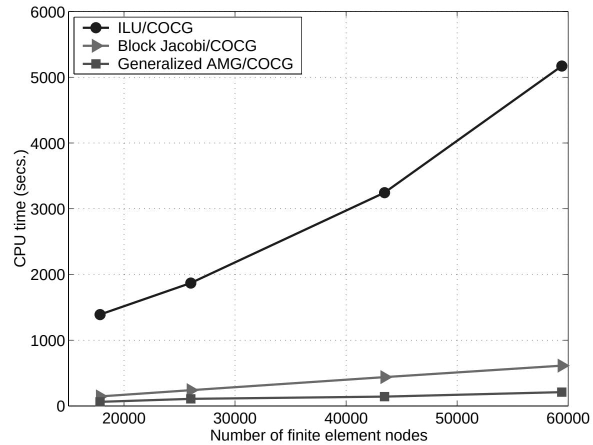

Fig. 1. Timings of multigrid and ILU preconditioned conjugate gradient method for complex symmetric systems.

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