Higher Order Material Spaces to Represent Non-Linear Material Characteristics and Source Terms in FEM-Computations

Johan Driesen, *Member, IEEE*, Hans Vande Sande, *Member, IEEE*
and Kay Hameyer, *Senior Member, IEEE*

***Abstract*-- A new approach to represent the local change of material properties or source terms due to their non-linear characteristics is presented here. The material function is given in terms of simple basis functions, allowing a higher order representation within the area covered by the finite element. This approximation as a higher order material space has advantages for the treatment of saturable ferromagnetic materials and temperature dependencies of conductive and permanent magnetic materials. It is possible to treat the material function space and the field solution as a coupled problem.**

***Index Terms*-- electro-thermal effects, ferromagnetic materials, finite element method, nonlinear differential equations**

I. Introduction

MANY partial differential equations describing electromagnetic or other physical fields often contain non-linear dependencies of the source term or material constants. These non-linearly dependent coefficients are usually scaled material characteristics, such as saturable ferromagnetic reluctivities or temperature dependent parameters, such as electric conductivities or permanent magnet material data. In the latter, the magnetic field equation can be a part of a larger electromagnetic-thermal coupled problem.

The non-linearities have to be treated very carefully, as the non-linear solution cannot be found sufficiently correct if these parameters are not accurately approximated. The other way around, the non-linearities cannot be determined if the solution is not achieved accurately because it depends on them and vice versa.

Non-linear dependencies can be considered in numerical field computation models, such as the finite element method (FEM), but this has to be done with care, especially in case of coupled problems, where multiple solution algorithms are involved.

Manuscript received June 3, 2000.

The authors are grateful to the Belgian "Fonds voor Wetenschappelijk Onderzoek Vlaanderen" for its financial support of this work and the Belgian Ministry of Scientific Research for granting the IUAP No. P4/20 on Coupled Problems in Electromagnetic Systems. The research Council of the K.U.Leuven supports the basic numerical research. J.Driesen is post-doctoral research fellow of the Belgian "Fonds voor Wetenschappelijk Onderzoek - Vlaanderen".

Johan Driesen, Hans Vande Sande and Kay Hameyer are with the Katholieke Universiteit Leuven, Dep. EE (ESAT), Div. ELEN, Kasteelpark Arenberg 10, B-3001 Heverlee, Belgium (tel.: +32.16.32.10.20, e-mail: johan.driesen@esat.kuleuven.ac.be).

II. Non-Linear Coefficient Treatment

A. Accuracy Issues

A general non-linear partial differential equations, with generic coefficients α, β, λ and f, for a field x, which could be an electrical, magnetic or thermal field, has the form:

$$\alpha(x)\frac{\partial x}{\partial t} - \nabla\cdot(\lambda(x)\nabla x) + \alpha(x)v\cdot\nabla x + \beta(x)x = f(x) \tag{1}$$

The FEM computes the solution of (1) as an approximation using polynomial base functions N_k:

$$\bar{x} = \sum_{k=1}^{m} N_k x_k \ . \tag{2}$$

The approximation error can be expressed as a function of an element size parameter h and the used polynomial order m [1]-[3]:

$$x = \bar{x} + O\left(h^{(m+1)}\right). \tag{3}$$

In the derivation of the algebraic FEM equations, integral expressions containing possibly dependent coefficients and a shape function expression are to be computed. For example, for the absorption term in (1), when the Galerkin method is used, this integral is:

$$\int_{\Omega_e} (\beta N_l \ N_k)\mathrm{d}\Omega \approx \tilde{\beta}\int_{\Omega_e} (N_l \ N_k)\mathrm{d}\Omega \,. \tag{4}$$

In the FEM, it is often assumed that the non-linear parameter β can be approximated as constant within the element's domain. Mathematically, this is identical to using zero-order elements to discretize the material or coefficient field:

$$\beta = \bar{\beta} + O(h) \tag{5}$$

The resulting accuracy for this term is:

$$\beta x = \overline{\beta}\overline{x} + \overline{x}_i O(h) + \overline{\beta} O\left(h^{(m+1)}\right) + \ldots \quad (6)$$

Therefore, the error of the product is of order one, no matter how high the polynomial order *m* of the element is.

It can be proved that the non-linear materials should be represented with an order of at least *m*-1 to maintain a similar approximation accuracy for (6) as in (3).

Similar considerations can be made regarding the other products in the diffusion, transient, convection and the force term of (1).

B. Complete Numerical Integration

It is theoretically possible to compute the integral (4) in a numerical way, for example by using Gauss points, with the non-linear coefficient included. In that case, it has to be evaluated at certain locations in the mesh. The order of the integration method has to be sufficient to obtain the required accuracy.

This approach may be complicated in the case of particular coupled problem calculations. Complicated large coupled problems are often computed as sequential procedures (e.g. [4]). Then, the field computations, for instance the magnetic and thermal field and the intermediate evaluations to determine the interactions (loss calculations and material parameter corrections) are executed in successive steps. Often, these subroutines use black-box solvers.

The practical implementation of a numerical integration method may then pose problems as the evaluation of the non-linearities requires the interpolation of the associated solution. Hence, the associated dicretisations and the subfield solutions have to be accessible at the same time.

III. Material Spaces Approach

A. General derivation

Alternatively, the coefficients/materials can be represented with a higher (non-zero) order polynomial approximation, which is calculated separately from the procedure for the field computation.

Hence, an entire finite element representation for the coefficients in (1) is constructed. For the previously used absorption coefficient, this is:

$$\overline{\beta} = \sum_{k=1}^{m^\beta} N_k^\beta \beta_k \ . \quad (7)$$

The functions N_k^β with the order of m^β can be of the same form as the finite element shape functions used in the actual FEM field solution. In the case of anisotropic materials represented by the coefficients, (7) can be extended. A finite element representation can be used for every material tensor entry.

If the approach (7) is used, the entire integral (4) becomes more complicated (one factor more in the products involving an additional shape functions expression), but it can still be calculated analytically. For example, assume first order triangular elements are used for both the field and material space representation:

$$N_k(x,y) = \frac{1}{2\Omega_e}\left(a_k + b_k x + c_k y\right), \quad (8)$$

with a_k, b_k, c_k a function of the element nodes' coordinates and Ω_e, the element's size [2]-[3]. Then, the matrix building block for the absorption term becomes:

$$\frac{\Omega_e}{10}\begin{bmatrix} \beta_1 + \frac{\beta_2}{3} + \frac{\beta_3}{3} & \frac{\beta_1}{3} + \frac{\beta_2}{3} + \frac{\beta_3}{6} & \frac{\beta_1}{3} + \frac{\beta_2}{6} + \frac{\beta_3}{3} \\ \frac{\beta_1}{3} + \frac{\beta_2}{3} + \frac{\beta_3}{6} & \frac{\beta_1}{3} + \beta_2 + \frac{\beta_3}{3} & \frac{\beta_1}{6} + \frac{\beta_2}{3} + \frac{\beta_3}{3} \\ \frac{\beta_1}{3} + \frac{\beta_2}{6} + \frac{\beta_3}{3} & \frac{\beta_1}{6} + \frac{\beta_2}{3} + \frac{\beta_3}{3} & \frac{\beta_1}{3} + \frac{\beta_2}{3} + \beta_3 \end{bmatrix}. \quad (9)$$

Similar element matrices can be derived for:

- the diffusion term with an anisotropic coefficient:

$$\frac{1}{12\Omega_e}\left(\left(\lambda_{1,x} + \lambda_{2,x} + \lambda_{2,x}\right)\left[b_i b_j\right] + \left(\lambda_{1,y} + \lambda_{2,y} + \lambda_{3,y}\right)\left[c_i c_j\right]\right), \quad (10)$$

- the convection term:

$$\frac{1}{12}\begin{bmatrix} \alpha_1 + \frac{\alpha_2}{2} + \frac{\alpha_3}{2} & \alpha_1 + \frac{\alpha_2}{2} + \frac{\alpha_3}{2} & \alpha_1 + \frac{\alpha_2}{2} + \frac{\alpha_3}{2} \\ \frac{\alpha_1}{2} + \alpha_2 + \frac{\alpha_3}{2} & \frac{\alpha_1}{2} + \alpha_2 + \frac{\alpha_3}{2} & \frac{\alpha_1}{2} + \alpha_2 + \frac{\alpha_3}{2} \\ \frac{\alpha_1}{2} + \frac{\alpha_2}{2} + \alpha_3 & \frac{\alpha_1}{2} + \frac{\alpha_2}{2} + \alpha_3 & \frac{\alpha_1}{2} + \frac{\alpha_2}{2} + \alpha_3 \end{bmatrix}\left[v_x b_i + v_y c_j\right], \quad (11)$$

- the force term:

$$\frac{\Omega_e}{6}\begin{bmatrix} f_1 + \frac{f_2}{2} + \frac{f_3}{2} \\ \frac{f_1}{2} + f_2 + \frac{f_3}{2} \\ \frac{f_1}{2} + \frac{f_2}{2} + f_3 \end{bmatrix}. \quad (12)$$

These matrix building blocks are symmetric (except for the convection term matrix).

It can be noticed that in this higher order material representation, the coefficients/materials have a similar representation, and equal 'status', as the field solution. The materials are represented in true 'material (vector) spaces'. Hence, the notion 'coupled physical fields' needs to be extended to 'coupled physical and material fields'. The determination of the weights in (7) has to be considered as a subproblem of the enlarged coupled problem at the same level as the involved physical fields.

The optimal combination of the polynomial orders of the solution and material approximation for a given field equation can be determined using expressions such as (6) for every term, taking into account that differentiation lowers the order by one.

B. Determination of the Material Space Coefficients

To assess the non-linearity in the iterative solution algorithm, whether it is due to a problem-own or a coupled problem dependency, several methods are possible:

1) Interpolation

The non-linearities are assessed using the field solutions. This is an explicit operation.

$$\beta_k = \beta\left(x\left(x_k, y_k\right)\right) \tag{13}$$

2) Least-squares method or Galerkin approach

The quadratic error between the material field and its polynomial approximation is minimized:

$$\min\left\{\int_{\Omega_e}\left(\beta - \sum_{k=1}^{m^\beta} N_k^\beta \beta_k\right)^2 d\Omega\right\}, \tag{14}$$

yielding:

$$\int_{\Omega_e} N_k^\beta \left(\beta - \sum_{k=1}^{m^\beta} N_k^\beta \beta_k\right) d\Omega = 0\,. \tag{15}$$

This equation is the same as would be obtained using a weighted residual approach with the Galerkin choice for the weighting functions.

This approach requires a (sparse) system solution. The size of this system depends on the number of elements containing non-linear materials.

IV. Application: Coupled Electromagnetic-Thermal Problems

A. Coupled Problem Computation

As an example, the practical implementation of this approach for a magnetic field computation is discussed. The 2D transient magnetic field modeled by means of the magnetic vector potential *A* [2], is described by:

$$\nabla\cdot\left(\upsilon(A)\nabla A\right) - \sigma(T)\frac{\partial A}{\partial t} = -\sigma(T)\Delta V_s - \upsilon(A)\nabla M\left(A,T\right), \tag{16}$$

and in the frequency domain using the time-harmonic method (in the frequency domain, the static magnetization *M* is omitted):

$$\nabla\cdot\left(\upsilon(\underline{A})\nabla\underline{A}\right) - \sigma(T)\,j\omega\underline{A} = -\sigma(T)\Delta\underline{V}_s\,. \tag{17}$$

The involved non-linear material coefficients are:

- ferromagnetic materials (which have to be modeled as dependent on the temperature at higher temperatures);
- electrical conductivities that depend on the temperature;
- permanent magnet characteristics that depend on the temperature and the magnetic field.

These are approximated as:

$$\upsilon(A) = \sum_{k=1}^{m^\upsilon} N_k^\upsilon \upsilon_k(A) \tag{18}$$

$$\sigma(T) = \sum_{k=1}^{m^\sigma} N_k^\sigma \sigma_k(T) \tag{19}$$

$$M(A,T) = \sum_{k=1}^{m^M} N_k^M M_k(A,T) \tag{20}$$

The thermal field equation is:

$$\nabla\cdot\left(\lambda(T)\nabla T\right) - \rho c(T)\frac{\partial T}{\partial t} = -q_{Joule}(T,A) \tag{21}$$

Here, the Joule loss is a function of both fields, as it contains the temperature dependent resistivity and the magnetic field invoking eddy currents:

$$q_{\mathrm{Joule},\Omega_e} = \frac{1}{\Omega_e}\int_{\Omega_e}\left(\sum_{k=1}^{m^\sigma} N_k^\sigma \sigma_k\right)\left(V_\mathrm{s} + \sum_{k=1}^{m^A} N_k^A \frac{\partial A_k}{\partial t}\right)^2 \mathrm{d}\Omega\,. \tag{22}$$

The non-linear thermal coefficients, including the loss density, are approximated in a similar way as in (18)-(20).

The calculation of this large coupled problem is performed using a block Gauss-Seidel approach [5] outlined in Fig. 1. A steady-state problem consists of one iteration loop; in the case of a transient problem, an extra time loop is added.

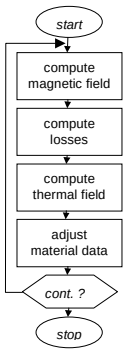


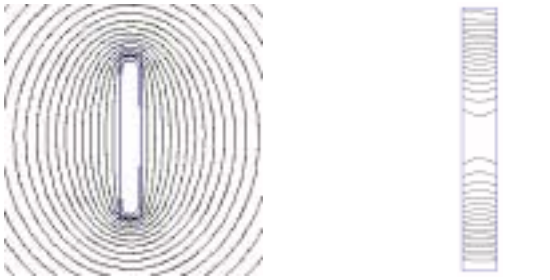
Fig. 1. Flowchart of a coupled electromagnetic-thermal computation using a block Gauss-Seidel method.

B. Simulation of Conductive Heating

To compare with results using higher-order approximations, the steady-state coupled electromagnetic-thermal solution of a

busbar is computed using the traditional FEM methodology with zero-order material coefficients.

The problem consists of a rectangular conductor with skin effect. It is cooled by means of natural air convection, which explains the necessity to use different meshes as the air region only needs to be meshed for the magnetic field computation. The field solutions are plotted in Fig. 2.



magnetic field thermal field

Fig. 2. Magnetic and thermal field solution.

The obtained zero- and first-order loss distributions are shown in Fig. 3 and Fig. 4. The continuous character of the first order distribution obviously gives a better detail in the skin effect region, which leads to a better calculation of the hot spot temperature present on that location.

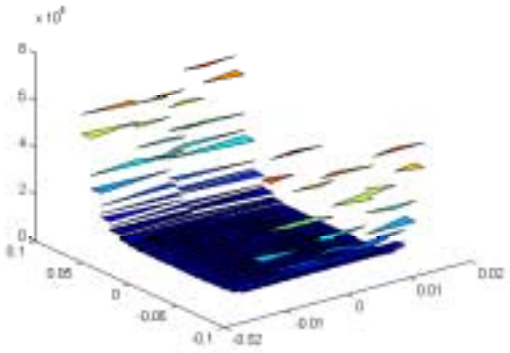


Fig. 3. Loss distribution in W/m2 using zero-order approximation.

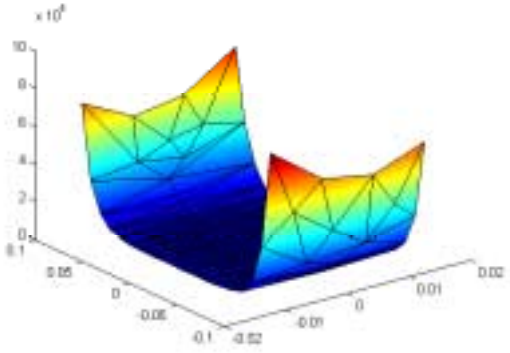


Fig. 4. Loss distribution in W/m2 using first-order approximation.

The evolution of the relative 2-norm convergence criterion is compared in Fig. 5. The methodology using the higher-order coefficient approximations converges faster than the zero-order approaches, even after refinement of the mesh. Furthermore, this methodology has an incremental change in computation time to solve an individual magnetic and thermal subproblems.

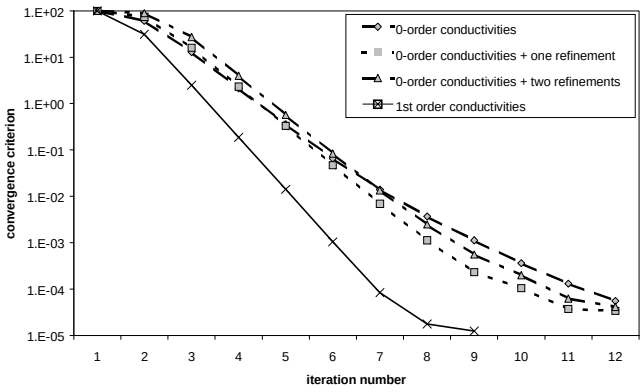


Fig. 5. Convergence criterion evolution for different approaches.

V. Conclusion

The requirement of employing accurate approximations for non-linear material coefficients, for instance featuring as a dependent material parameter, is demonstrated for a general field equation. An alternative to approximating the material coefficient to be constant within the element, implementing higher-order 'material spaces' is outlined here. It uses finite element approximations for the non-linear parameters. The extended building blocks for the finite-element field equations are derived. Non-linear updating procedures, explicit interpolation and a least-squares fitting, are discussed.

The implementation and advantages of this approach in a coupled electromagnetic-thermal problem are illustrated.

References

[1] C. Johnson, Numerical solution of partial differential equations by the finite element method, Cambridge University Press, 1987.

[2] P.P. Silvester, R.L. Ferrari, Finite elements for electrical engineers, 3rd ed., Cambridge University Press, 1996.

[3] K.J. Binns, P.J. Lawreson, C.W. Trowbridge, The Analytical and Numerical Solution of Electric and Magnetic Fields, Wiley, 1992.

[4] J.Driesen, G.Deliége, K.Hameyer, "Coupled thermo-magnetic simulation of a foil-winding transformer connected to a non-linear load," to be published in *IEEE Trans. on Magnetics*, 4p.

[5] J.Driesen, K.Hameyer, "Comparison of Strong and Weak Coupled Solution Algorithms for Coupled Electromagnetic-Thermal Problems," accepted for presentation at CEFC 2000, Milwaukee, USA, 4-7 June 2000.