A Mathematical Framework for the Finite Element Modelling of Electromechanical Problems

F. Henrotte and K. Hameyer *

K.U.Leuven, Dept. ESAT, Div. ELEN,
94, Kardinaal Mercierlaan, B-3001, Leuven, Belgium

Abstract Besides differential forms, special tensor fields are introduced and discussed for the representation of physical fields in continuous medium problems.

Introduction

When adressing coupled problems, experience shows that it is worthwhile to avoid notions that are specific to a particular branch of Physics (e.g. vectors in electromagnetism and tensors in elasticity) for the benefit of more abstract mathematical notions. These unified notions are less restrictive and more richly endowed with mathematical properties. They are thus liable to be implemented as properly designed objects with an *object-oriented* programming language in a *finite element* programme. On the way, one has been led to borrow notions from different mathematical disciplines. Conservation laws can be written in a metric-independent manner, thanks to *differential geometry* notions. Also, *group theory* and *functional analysis* are appropriate tools for tackling with the constitutive laws.

1 Differential geometry for Electromagnetism

Manifold: A *manifold* M of dimension n is a continuous set of points of which any neighborhood can be mapped by a differentiable 1-1 mapping onto a subset of $\mathbb{R}^n$. By assuming only the *existence* of the mapping, without requiring it to be explicitly specified, the manifold is endowed with a *differentiable structure* (it actually inherits that of $\mathbb{R}^n$) without being equipped with a preferred global co-ordinate system or even the notion of *distance*. These notions will be introduced at the right time in Sect. 3, leaving thus a substantial part of the theory foreign to any consideration of measure or distance. This will turn out to have great computational advantages.

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Vectors, covectors and tensors: The simplest object one can build on a manifold is a parametric *curve*, i.e. a differentiable rule $t \in]t_A, t_B[\subset \mathbb{R} \mapsto M$. The mathematical quantities that may rightly be called *vectors* are the tangent vectors to such curves. The set of all vectors that are tangent to curves going through a point $P \in M$ forms a *linear space* of dimension n. This result, which is true whatever the complexity of M itself, comes merely from the definition of a manifold and the fact that partial derivatives are linear operators in $\mathbb{R}^n$. This linear space is called *tangent space* and noted $T_P(M)$.

A *covector* a is defined as a real-valued linear operator on vectors:

$$a : v \in T_P(M) \mapsto a(v) \in \mathbb{R}. \tag{1}$$

The set of all covectors at point P also forms a linear space of dimension n that is called *cotangent space* and noted $T_P^*(M)$. Generalizing further by means of the tensor product $\otimes$, a *tensor* A is defined as a real-valued multilinear operator on ordered sets of vectors and covectors

$$A : v, \dots \in T_P(M),\ a, \dots \in T_P^*(M) \mapsto A(v, \dots; a, \dots) \in \mathbb{R}. \tag{2}$$

Tensors with p vector arguments and q covector arguments form a linear space of dimension n^{p+q}. In a co-ordinate system $\{x^i, i = 1, \dots, n\}$, any tensor admits an expansion

$$A = A_{i \dots j}^{k \dots l}\, dx^i \otimes \dots \otimes dx^j \otimes \frac{\partial}{\partial x^k} \otimes \dots \otimes \frac{\partial}{\partial x^l} \tag{3}$$

where the dx^k's (i.e. the gradients of the co-ordinates) form a convenient basis for covectors and the $\frac{\partial}{\partial x^k}$'s (i.e. the vectors tangent to co-ordinate lines) form a convenient basis for vectors. A *tensor field* is a rule that associates a tensor to each point of M.

Integration and p–forms: What is expected from a tensor field A for being an argument for a real-valued p–*integral*

$$I = \int_\Omega A \tag{4}$$

where Ω is a domain of dimension p? To answer this question, an infinitesimal piece $\Delta\Omega$ of a p–dimensional domain Ω is considered. It can be seen as being spanned with a set of p linearly independant infinitesimal vectors. It is also required that its orientation changes either by reversing one vector or swapping two of them. This amounts to saying that $\Delta\Omega$ must be proportional to the completely antisymmetrised tensor product of a set of p vectors. This is called a p–*vector*. The assertion is actually general: *antisymmetrisation* is the operation that selects the particular tensors that play a role for integration.

Now, the argument of a p–integral is expected to associate a real value (the local infinitesimal contribution to the integral) to each infinitesimal piece of Ω. It is thus defined as a real-valued linear operator on p–vectors or, equivalently, a tensor with p vector-arguments, that has been particularised in order that $A(.., v, .., w, ..) = -A(.., w, .., v, ..)$ holds for any pair of arguments. This is called a p–*covector*. In a 3D space, antisymmetrisation only allows non-zero p–covectors for $p = 0, 1, 2, 3$.

Fields of p–covectors are (by construction) arguments for p–integrals. They are called p–*form*. They respectively span linear spaces of dimension $C_p^n = \frac{n!}{(n-p)!p!}$ that are noted $F^p(M)$, $p = 0, 1, 2, 3$. They need to be handled by specific intrinsic antisymmetry-preserving operators. The antisymmetry-preserving tensor product is the *exterior product* $\wedge : F^p(M) \times F^q(M) \mapsto F^{p+q}(M)$ that verifies

$$\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha. \tag{5}$$

The antisymmetry-preserving derivative is the *exterior derivative* $d : F^p(M) \mapsto F^{p+1}(M)$ that obeys *Leibniz rule* (6) and *Stokes theorem* (7)

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^p \alpha \wedge d\beta \tag{6}$$

$$\int_\Omega d\alpha = \int_{\partial\Omega} \alpha \tag{7}$$

where $\partial\Omega$ is the *boundary* of $\Omega \subset M$. Another fundamental differential operator is the *Lie derivative* $\pounds_{\frac{d}{dt}}$ that allows us to compute time derivatives of integrals over moving and deforming domains. One has

$$\partial_t I(t) = \partial_t \int_\Omega A = \int_\Omega \pounds_{\frac{d}{dt}} A \tag{8}$$

where $\frac{d}{dt}$ stands for the bundle of trajectories (curves) generated by the movement of each point of Ω. Complete definitions of the notion presented in this section can be found in any Differential geometry treatise [?,2].

Conservation laws in Electromagnetism: *Electromagnetic fields* are paradigms for p–forms. If the magnetic field h and the electric field e are represented by 1–forms; the induction field b, the electric displacement d and the current density j by 2–forms; the charge density ρ^Q by a 3–form, one can write with the notions introduced so far the *Poynting relation* (9) and *charge conservation* (10) that are the *global conservation principles* electromagnetic fields must obey:

$$\int_\Omega (\partial_t b \wedge h + e \wedge \partial_t d) + \int_\Omega e \wedge j - \int_{\partial\Omega} e \wedge h = 0, \tag{9}$$

$$\partial_t \int_\Omega \rho^Q + \int_{\partial\Omega} j = 0. \tag{10}$$

As they must hold for all e, h and Ω, their famous *local* forms immediately follow from (7), (6) and (8):

$$\begin{cases} de + \partial_t b = 0 \\ dh - \partial_t d = j \\ dj + \pounds_{\frac{d}{dt}} \rho^Q = 0 \end{cases} \tag{11}$$

2 Continuous medium mechanics

Vector and covector valued p−forms: Whereas p−forms are the tensors that perfectly suit the electromagnetic fields, other kinds of tensors have to be selected in order to represent adequately the mechanical tensors (stress and strain).

A tensor α with $p+1$ vector arguments but that has been antisymmetrised only for its first p arguments is called a *covector valued p−form*. Similarly, a tensor β with p antisymmetrised vector arguments and one covector argument is called a *vector valued p−form*. Both have nC_n^p linearly independent components. They are noted

$$\alpha = \alpha_k \otimes dx^k \in F^{p\oplus 1}(M) \quad , \quad \beta = \beta^k \otimes \frac{\partial}{\partial x^k} \in F^p_{\oplus 1}(M) \tag{12}$$

where the α_k's and the β^k's are n p−forms.

As the exterior product $\wedge$ and the exterior derivative d are defined on p−forms only, they may operate on the α_k's and the β^k's but need to be generalised for tensors like α and β. One *defines*, $\forall \alpha \in F^{p\oplus 1}(M)$, $\forall \beta \in F^q_{\oplus 1}(M)$,

$$\alpha \wedge \beta = (\alpha_i \otimes dx^i) \wedge (\beta^j \otimes \frac{\partial}{\partial x^j}) = \alpha_i \wedge \beta^i, \tag{13}$$

$$d\alpha = (d\alpha_i) \otimes dx^i, \; d\beta = (d\beta^j) \otimes \frac{\partial}{\partial x^j}, \tag{14}$$

so that the Leibniz rule (6) still holds. [1]

Mechanical tensors: We are now ready to present the tensor structure of the mechanical tensors. Let $\frac{d}{dt} \in F^0_{\oplus 1}(M)$ be a *velocity field* on Ω, i.e. the field of the vectors tangent to the bundle of trajectories generated by the movement of the points of Ω. As a force f is a covector, a *volume density of force* ρ^f is a volume density of covector, i.e. $\rho^f \in F^{3\oplus 1}(M)$, of which the developed *mechanical power density* is, by (13), $\rho^f \wedge \frac{d}{dt} \in F^3(M)$. Similarly, a *stress tensor* T is a surface density of force, i.e. $T \in F^{2\oplus 1}(M)$, developing $T \wedge \frac{d}{dt} \in F^2(M)$. The (linear) *momentum density* $\rho^p \in F^{3\oplus 1}(M)$ has the same tensor structure as ρ^f and the density of *kinetic energy* is $\rho^p \wedge \frac{d}{dt} \in F^3(M)$.

The *global conservation law for momentum* is

$$\partial_t \int_\Omega \rho^p \wedge \frac{d}{dt} + \int_\Omega T \wedge D = \int_\Omega \rho^f \wedge \frac{d}{dt} + \int_{\partial\Omega} T \wedge \frac{d}{dt}. \tag{15}$$

[1] It may rightly be objected that (13) and (14) assume the implicit definition of a particular connection on the manifold which, moreover, relies on the coordinates. This lack of generality is certainly a theoretical shortcoming of this approach but it allows considerable simplifications in the algebraic developpements and for the implementation. It also makes the parallelism between the electromagnetic and mechanic systems more clear. As, from a theoretical point of view, a connection must be defined in any case and engineers are not used to such a notion, this implied introduction of the most customary connection has a virtue of simplicity.

wherein the mechanical power density developed by the stresses has been noted $T \wedge D \in F^3(M)$ with $D \in F^1_{\oplus 1}(M)$ the energetic dual of T. As (15) must hold for all $T, \frac{d}{dt}, \Omega$, it has the *local* form

$$\begin{cases} \pounds_{\frac{d}{dt}} \rho^p = dT + \rho^f \\ D = d(\frac{d}{dt}) \end{cases} \tag{16}$$

where the first equation is *Newton's law* and the second one determines D as being the *gradient of the velocity field.* The striking similarity of (9,15) for the global equations and (11,16) for the local ones was already pointed out by Tonti in [3].

Electromagnetic forces: A first advantage derived from the definition of theses new kind of tensors is that they allow a much easier expression of the **electromagnetic forces** f_{EM}. Indeed, thermodynamical considerations allow to define them via the **virtual work principle** [4] by

$$\int_\Omega \rho^{f_{EM}} \wedge \frac{d}{dt} = \int_\Omega \pounds_{\frac{d}{dt}} \rho^{\Psi_{EM}} \text{ with } \pounds_{\frac{d}{dt}} b = 0 \tag{17}$$

where $\frac{d}{dt}$ is a virtual velocity field and $\rho^{\Psi_{EM}}$ is the density of electromagnetic energy, which is a function of the induction field b.

3 Constitutive Laws

The evolution of a physical system is not fully determined by conservation laws. *Constitutive laws* are also necessary to express the behavior of the matter involved. In a finite element programme, especially when multiphysic interactions are considered (e.g. electromechanical), the constitutive laws are advantageously represented by *energy functionals* in order to track as closely as possible the thermodynamics of the interaction. An energy functionals is a rule that associates a certain amount of energy to a field. This requires therefore to attribute a certain idea of *intensity* to the field. As, for a given flux through a surface, the intensity of the field depends on the size of the surface, defining an intensity amounts to the definition of a *metric* on M, i.e. the notion of distance. Practically, the metric g is a symmetric tensor with two vector-arguments, i.e. $g(v, w) = g(w, v)$, which allows to define a *norm* $|x|$ for the tensors. Together with a *volume form* $\pi \in F^3(M)$, it also allows to define the *Hodge operator*,

$$* : F^p(M) \mapsto F^{n-p}(M) \tag{18}$$

$$*\alpha = (*\alpha_k) \otimes dx^k, *\beta = (*\beta^k) \otimes \frac{\partial}{\partial x^k}. \tag{19}$$

Discussion: This paper stays at a modest level with regard to constitutive laws, i.e. linear elasticity, infinitesimal displacements. It is only shown that the Hooke law becomes a scalar one, and no more a tensorial one, in the group structure of the tensor sets we have defined.

The tensors T and D appearing in (16) are not the classical σ and ε tensors. This is shown by analysing the group structure of the 9-dimensional linear spaces they span, which are

$$D \in F^1_{\oplus 1}(M) \overset{g}{\longleftrightarrow} D^\flat \in F^{1\oplus 1}(M) \overset{*}{\longleftrightarrow} T \in F^{2\oplus 1}(M) \qquad (20)$$

where the intermediary space is $D^\flat = D^k_i g_{kj} dx^i \otimes dx^j \in F^{1\oplus 1}(M)$. By a decomposition of these three spaces into irreducible subspaces, one can split each of them into a subspace of dimension 3 with *rotation* modes (indexed R) and a subspace of dimension 6 with *deformation* modes (indexed D). In the latter, a subspace of dimension 5 with *shear* modes (indexed S) and a subspace of dimension 1 with *compression* modes (indexed C) may be further distinguished. With $tr_g T = T_{ijk} g^{jk} dx^i \in F^1(M)$, this can be summarised as follows:

		$D^\flat \in F^{1\oplus 1}(M)$	$T \in F^{2\oplus 1}(M)$
R		$D^\flat$ is antisym.	$tr_g T \neq 0$
D	S	$D^\flat$ is sym.	$tr_g T = 0$
	C	$D^\flat \propto g$	$T \propto \pi$

(21)

One can see that T and D have an antisymmetric part that σ and ε have not. This relaxation is the condition *sine qua non* to have the linear momentum conservation law (15) expressed in a metric-independent form.

The group decomposition (21) allows to state the constitutive law of an elastic medium in a very simple way. The physical idea of *deformation* is related with $D^\flat_D = \frac{1}{2}\pounds_{\frac{d}{dt}} g$ [5] where D_R plays no role because infinitesimal rotations do not participate in the stored elastic energy. One has now for the *Hooke law* of a linear isotropic material

$$T_C = 3K \ * D^\flat_C, \quad T_S = 2G \ * D^\flat_S, \quad T_R = 0, \qquad (22)$$

where K is the *bulk modulus* and G the *shear modulus*. The third relation is equivalent to the classical $\sigma_{ij} = \sigma_{ji}$ in orthonormal co-ordinates. It restores then, at the constitutive law level, the due symmetry of the stress tensor but it also makes the constitutive law *injective*, i.e. the knowledge of T does not fully determine D. Such constitutive laws can be properly handled by *Convex analysis* [6] which associates to the elastic energy functional directly derived from (22), i.e.

$$\Psi(D) = \frac{1}{2}\int_\Omega (3K\,|D_C|^2 + 2G\,|D_S|^2)\pi, \qquad (23)$$

its *Legendre transform* where D_R plays the role of a Lagrange multiplier for the $T_R = 0$ equation, i.e.

$$\Phi(T, D_R) = \frac{1}{2}\int_\Omega (\frac{|T_C|^2}{3K} + \frac{|T_S|^2}{2G})\pi - \sup_{D_R}\int_\Omega T_R \wedge D_R. \qquad (24)$$

Conclusion

A finite element formulation of a continuous medium problem relies on a *minimum principle* which itself relies on a *function space* F and a *functional* Λ defined on F. Provided the fields in presence are represented by $p-$forms or by (co)vector-valued $p-$forms, they can be interpolated with *Whitney forms* (special shape functions that ensure an interpolation that are always conforming and that preserve the right continuity properties at the discrete level [7,8]). Moreover, global and local conservation laws in electromagnetism and elasticity can be expressed as metric-independent differential relations in a strikingly similar manner. These are the main achievements of this paper. There are practical implications. A great part of the desirable properties of the Whitney forms that are exploited in electromagnetic computations can be translated straightfully into equivalent techniques for elasticity computations. This helps a lot for the implementation of an electromechanical finite element programme and to the understanding of the difficult problem of electromagnetic forces.

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