

Strong Coupled Multi-Harmonic Finite Element Simulation Package

Herbert De Gersem, Hans Vande Sande, Kay Hameyer

Abstract— The harmonic balanced finite element method offers a valuable alternative to the transient finite element method for the quasi-static simulation of electromagnetic devices operating at steady-state. The specially designed iterative solver, the adaptive relaxation of the non-linear loop and the embedding of the harmonic balanced finite element method within a state-of-the-art finite element package, leads to a solver in the frequency domain that is competitive to time stepping. The benefits of this approach are illustrated by its application to an inductor with a ferromagnetic core.

Keywords— Finite element methods, Harmonic analysis, Magnetic cores.

I. INTRODUCTION

QUASI-STATIC finite element (FE) simulation of electromagnetic energy transducers operating at steady-state, has a large technical relevance but is still challenging, especially if ferromagnetic materials, induced currents and motional effects are involved [1]. These phenomena introduce time dependencies at different time scales into the electromagnetic behaviour of the device. A transient solver applied to resolve the variation in time of the field quantities, relies upon explicit time stepping requiring considerable simulation times [2]. The alternative time-harmonic approach applies the transform of the partial differential equation (PDE) in the frequency domain and provides the FE solution at one frequency in a single computational step [3]. As the latter is not sufficient to capture all relevant phenomena occurring in many technical devices, Yamada, Bessho and Lu proposed the Harmonic Balanced Finite Element Method (HBFEM), incorporating a larger set of harmonic components in the FE simulation [4]. Non-linear effects and moving parts, present in the model, are responsible for a strong coupling between the components at different frequencies. HBFEM yields a periodic solution in one single step. This advantage is partially annulled by the considerably higher complexity of the system of equations. Calculating in the frequency domain is both at the level of understanding as from the computational point of view, cumbersome. In this paper, a novel implementation of HBFEM is proposed. A careful implementation of the material data and a specially designed iterative solver for the system of equations, improve the simulation speed and again highlights the benefits of HBFEM over transient simulation in the case of steady-state simulation. This is illus-

trated by the application of HBFEM to a model of technical relevance.

II. TRANSIENT OR HBFEM SIMULATION

The criteria considered to select a transient or HBFEM simulation, are collected in Table I. For quasi-statics, transient computation is obviously the most general approach. It captures all harmonic components as long as a sufficiently small time step is applied [5]. A large number of time steps applied to a complicated model, may however cause an unnecessary burden on the simulation. If only a few a-priori known harmonic components of the steady-state behaviour are searched for, a HBFEM solver is more appropriate [6]. The inconvenience of the a-priori selection of the set of frequencies involved in the HBFEM simulation, can be circumvented by making this choice adaptive within the steps of the non-linear loop. If additionally, a slow varying transient component has to be considered, the envelop approach can be applied [7]. The main drawback of HBFEM is the size and complexity of the system of equations corresponding to strongly combined discretisations of the spatial and the frequency domains [8]. If the size exceeds the available memory and/or hampers the convergence of the iterative solver, this may be a reason to turn over to a transient simulation.

III. FERROMAGNETIC SATURATION

Ampère's law defines a linear relationship between the electric current $i(t)$ and the magnetic field strength $h(t)$. As a consequence, their harmonic patterns are identical (Fig. 1). The same is true for the electric voltage $v(t)$ and the magnetic flux density $b(t)$, related to each other by the law of Faraday-Lenz. The relation of the magnetic field strength to the magnetic flux density is, however, determined by a constitutive law, defining the behaviour of the material when submitted to a magnetic field. The non-linearity of the BH -characteristic, e.g. in the case of ferromagnetic saturation, causes the introduction of and the coupling between all harmonic components present in both field quantities. The relation between the electric quantities depends upon the conductivity of the material and the circuit elements, which are both assumed to be linear. As a consequence, a coil exciting a magnetic core with a pure sinusoidal current, experiences a voltage with additional higher harmonic components due to the non-linear magnetisation characteristic. A harmonic signal applied to the magnetisation characteristic yields in general besides the constant term, an infinite series of harmonics with double harmonic orders. Hence, if the excitation $i(t)$ only consists

H. De Gersem, H. Vande Sande and K. Hameyer are with the Katholieke Universiteit Leuven, Dep. ESAT, Div. ELEN, Kardinaal Mercierlaan 94, B-3001 LEUVEN, Belgium. Phone: +32-16-321020; Fax: +32-16-321985; E-mail: Herbert.DeGersem@esat.kuleuven.ac.be, Hans.VandeSande@esat.kuleuven.ac.be, Kay.Hameyer@esat.kuleuven.ac.be; WWW: <http://www.esat.kuleuven.ac.be/elen/elen.html>.

transient	HBFEM
+ captures all harmonics	- selected set of harmonics
- marching in time	+ solution in a single step
+ small FE systems	- large FE system
- troublesome if frequencies differ largely	+ advantageous for large differences in frequencies

of odd harmonics, the spectrum of the permeability $\mu(t)$ only contains even harmonics (Fig. 1). As the convolution of both again yields only odd harmonics and the coupling via the electric properties is linear, $b(t)$ and $v(t)$ are comprised of odd harmonics.

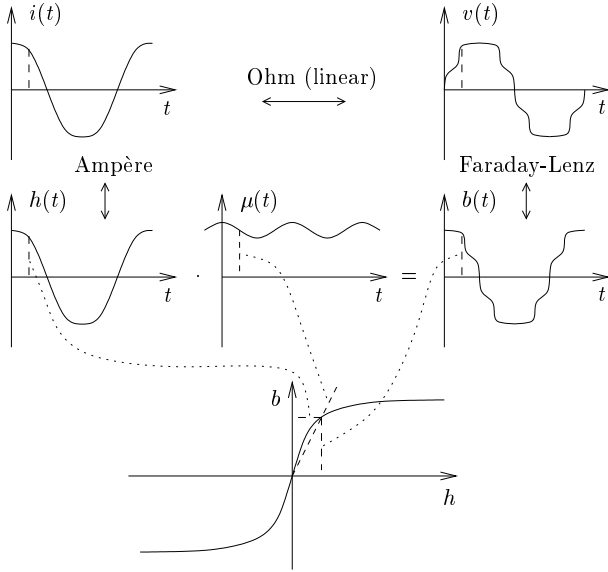


Fig. 1. Harmonic contents of the current $i(t)$, the magnetic field strength $h(t)$, the magnetic flux density $b(t)$, the voltage $v(t)$ and the permeability $\mu(t)$.

IV. FINITE DISCRETE SPECTRA

Both, the evaluation of the non-linear constitutive law and the convolution with the material spectrum, cause an expansion of the harmonic pattern. In simulation, only a limited set of harmonics is considered. In practice, only a few components of the permeability are computed and furthermore, a truncated convolution is applied (Fig. 2). The designer selects the orders of the harmonics admitted to the HBFEM simulation on the basis of technical considerations. In electric machinery, usually only a few harmonics are technically relevant. For particular models, this selection may be done adaptively with respect to intermediate solutions.

V. MAGNETODYNAMIC FINITE ELEMENT MODEL

A. Formulation

The condition $\nabla \cdot \mathbf{B} = \mathbf{0}$ for the magnetic flux density \mathbf{B} , is explicitly fulfilled by applying the magnetic vector potential \mathbf{A} , i.e. $\mathbf{B} = \nabla \times \mathbf{A}$. Faraday-Lenz's law for

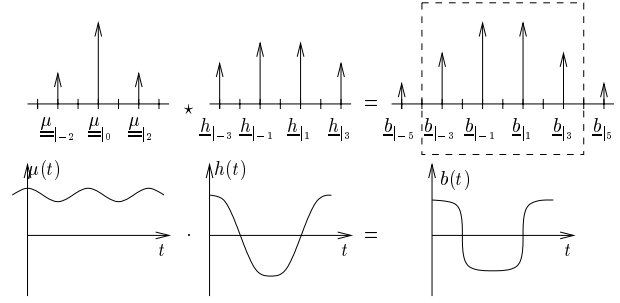


Fig. 2. Truncated convolution $\underline{\mu} \star \underline{h} = \underline{b}$.

the electric field \mathbf{E} is integrated up to the gradient of the electric voltage V :

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}. \quad (1)$$

The current density \mathbf{J} and the magnetic field strength \mathbf{H} are related to \mathbf{E} and \mathbf{B} respectively by the conductivity σ and the reluctivity $\nu = 1/\mu$. The governing equation is Ampère's law in terms of the magnetic vector potential,

$$\nabla \times (\nu \nabla \times \mathbf{A}) + \sigma \frac{\partial \mathbf{A}}{\partial t} = -\sigma \nabla V. \quad (2)$$

The model is excited by a voltage field V . The HBFEM formulation is obtained by transforming (2) into the frequency domain,

$$\nabla \times (\underline{\nu} \star \nabla \times \underline{\mathbf{A}}) + \underline{\sigma} \star \mathcal{T}(\underline{\mathbf{A}}) = -\underline{\sigma} \star \nabla V, \quad (3)$$

and discretising the latter as indicated in Sections III and IV. \star denotes convolution. \mathcal{T} is the operator representing the Fourier transform of the time derivative, i.e.

$$\mathcal{T} : g(\omega) \rightarrow j\omega g(\omega). \quad (4)$$

The applications considered here, allow for a 2D discretisation of the z -component of the magnetic vector potential,

$$\underline{A}_z = \sum_{j=1}^{n_f} \underline{A}_{zj} N_j(x, y), \quad (5)$$

in terms of n_f triangular and linear, finite elements N_i defined on the computational domain Ω . The application of the weighted residual method is similar to the one in the frequency domain. The resulting system of equations,

$$(\underline{K} \star + \underline{L} \star \mathcal{T}) \underline{x} = \underline{f}, \quad (6)$$

with

$$\underline{\underline{k}}_{ij} = \int_{\Omega} \underline{\underline{\nu}} \nabla N_i \cdot \nabla N_j d\Omega; \quad (7)$$

$$\underline{\underline{l}}_{ij} = \int_{\Omega} \underline{\underline{\sigma}} N_i N_j d\Omega; \quad (8)$$

$$\underline{\underline{x}}_j = \underline{\underline{A}}_{zj}; \quad (9)$$

$$\underline{\underline{f}}_i = \int_{\Omega} \underline{\underline{\sigma}} \star \nabla \underline{\underline{V}} N_i d\Omega, \quad (10)$$

involves matrix convolutions and the application of differentiation. Because $\underline{\underline{k}}_{ij} = \underline{\underline{k}}_{ji}$, $\underline{\underline{l}}_{ij} = \underline{\underline{l}}_{ji}$ and \mathcal{T} are a symmetric operators, (6) denotes a symmetric system. In the following, the conductivity is assumed to be linear. As a consequence, $\underline{\underline{\sigma}}$ and hence $\underline{\underline{l}}_{ij}$, simplify to the scalars σ and l_{ij} .

B. Real Equivalent Formulation

The system of equation can be represented by a real equivalent system of equations of dimension $2n_h n_f$ [4]. If e.g. 2 harmonics are considered, the convolution

$$(\underline{\underline{a}} + b\mathcal{T}) \underline{\underline{x}} = \underline{\underline{y}}, \quad (11)$$

denoted in matrix notation, is

$$\begin{bmatrix} a_0 & a_2^{re} & -a_2^{im} & 3b\omega \\ a_2^{re} & a_0 + a_2^{re} & a_2^{im} + b\omega & a_2^{im} \\ -a_2^{im} & a_2^{im} - b\omega & a_0 - a_2^{re} & a_2^{re} \\ -3b\omega & a_2^{im} & a_2 & a_0 \end{bmatrix} \begin{bmatrix} \underline{\underline{x}}_3^{re} \\ \underline{\underline{x}}_1^{re} \\ \underline{\underline{x}}_1^{im} \\ \underline{\underline{x}}_3^{im} \end{bmatrix} = \begin{bmatrix} \underline{\underline{y}}_3^{re} \\ \underline{\underline{y}}_1^{re} \\ \underline{\underline{y}}_1^{im} \\ \underline{\underline{y}}_3^{im} \end{bmatrix}. \quad (12)$$

It is clear that this formulation is not longer symmetric. Generalising this result to the system of equations (6), it is obvious that the symmetry of the magnetodynamic partial differential equation (3) is not reflected in its real equivalent discretisation.

C. Krylov Subspace Solver

In [8], it is shown that the iterative solution of (6) directly, is more advantageous than solving its real equivalent. The symmetry of the original system enables the application of the symmetric Lanczos algorithm, yielding solvers such as Conjugate Gradients or, in the case considered here, symmetric Quasi-Minimal Residual (QMR), instead of the computationally more expensive Arnoldi method or Lanczos biorthogonalisation, as applied in the Generalised Minimal Residual and the Bi-Conjugate Gradient type solvers respectively [9][10]. The direct implementation of QMR to (6) partially cures the main drawback of HBFEM, i.e. the excessive time spent to solve the linear system.

D. Non-Linear Loop

The non-linearity of the reluctivities of each element of the FE mesh, is resolved by a non-linear iteration.. The application of the Newton method is cumbersome because the formulation yields a non-analytical system of equations, as

stated for the special case of one frequency in [11]. The construction of the Jacobian would spoil the beneficial properties of (6). Instead, Picard iteration is applied. For the technical models under consideration, convergence is only achieved if substantial relaxation is invoked. If $\underline{\underline{x}}$ is the solution of the linearised system

$$\left(\underline{\underline{K}}^{(k)} \star + \underline{\underline{L}} \star \mathcal{T} \right) \underline{\underline{x}} = \underline{\underline{f}}^{(k)}, \quad (13)$$

a new approximative solution,

$$\underline{\underline{x}}^{(k+1)} = \gamma \underline{\underline{x}} + (1 - \gamma) \underline{\underline{x}}^{(k)} \quad (14)$$

is determined in terms of the relaxation factor γ . γ is selected out of the set 1, 0.5, 0.25, 0.125, 0.625, minimising the non-linear residual itself:

$$\| \underline{\underline{f}}^{(k+1)} - \left(\underline{\underline{K}}^{(k+1)} \star + \underline{\underline{L}} \star \mathcal{T} \right) \underline{\underline{x}}^{(k+1)} \|. \quad (15)$$

This involves the assembling of $\left(\underline{\underline{K}}^{(k+1)} \star + \underline{\underline{L}} \star \mathcal{T} \right)$ and $\underline{\underline{f}}^{(k+1)}$ for each tested value of γ . This procedure enters an expensive minimisation into the simulation. Its necessity, however, is affirmed by the numerical experiments upon various technical models.

VI. IMPLEMENTATION

The HBFEM solver presented here, is implemented within an existing FE software [12]. To benefit from the rich variety of already existing, advanced FE techniques, e.g. adaptive mesh refinement and field-circuit coupling, not only the original structure of the software but also the existing code is maintained. The HBFEM approach is incapsulated in the simulation tool by relying upon the operator overloading features offered by a programming language supporting object oriented software design, such as C++ [13][14].

First, all occurring variables are classified according to their harmonic content: real values, complex values, odd harmonic spectra and even harmonic spectra (Table II). This splitting requires a well-defined organisation of the software and introduces a more restrictive type checking when compared to the situation where only real values are considered. The advantage, however, is quite substantial: a whole bunch of FE solvers originates from the same coding by simply assigning appropriate types to these generic classes, as indicated in Table II, and recompiling.

Secondly, the meaningful operators operating upon a single or upon two arguments, are defined (Table III). Notice that the type of the result achieved by the operator depends upon the types of its arguments. The convolution of two odd spectra gives an energy related quantity and is within the software restricted to its DC-component, whereas the convolution of two even spectra is an even spectrum and the convolution of an even with an odd spectrum yields an odd one. Also here, the rigorous mathematical treatment offers particular benefits for conceiving and maintaining a neat software. The overloading principle selects the appropriate operator according to the types of its operands.

TABLE II

GENERIC TYPES (DENOTED BY TYPE), EXAMPLES AND THE ASSIGNED TRUE TYPES YIELDING DIFFERENT FE SOLVERS.

general type	examples	static solver	time-harmonic solver	HBFEM solver
type-real	lengths, radii	double	double	double
type-complex	energies	double	complex	complex
type-odd	currents, fluxes, voltages, field strenghts	double	complex	odd harmonic spectrum
type-even	permeabilities, conductivities	double	complex	even harmonic spectrum

TABLE III
GENERIC OPERATORS.

operator	1st argument	2nd argument	result	meaning
+/-	type-real	type-real	type-real	addition/subtraction
+/-	type-complex	type-complex	type-complex	addition/subtraction
+/-	type-odd	type-odd	type-odd	addition/subtraction
+/-	type-even	type-even	type-even	addition/subtraction
*	type-real	type-real	type-real	multiplication/convolution
*	type-complex	type-complex	type-complex	multiplication/convolution
*	type-odd	type-odd	type-complex	multiplication/convolution
*	type-even	type-even	type-even	multiplication/convolution
*	type-even	type-odd	type-odd	multiplication/convolution
*	type-real	type-complex	type-complex	multiplication
*	type-real	type-odd	type-odd	multiplication
*	type-real	type-even	type-even	multiplication

In a time domain solver, an * corresponds to multiplication whereas the same coding will activate a convolution operator when compiled into a frequency domain solver.

The odd and the even spectra, substituted for the type-odd and type-even values respectively while compiling a HBFEM solver, are considered as separate programming objects. Appropriate operators, such as addition, multiplication by a scalar, differentiation and convolution, are implemented as overloaded operators. The evaluation of a non-linear material characteristic and the application of an inverse convolution are expensive operators and burden the HBFEM solver. Both involve Fast Fourier Transforms (FFT). Two enhancements are introduced. For small spectra, FFT tends to be inefficient and is replaced by a modified version of the Levinson algorithm [15]. Secondly, the application of matrix-vector products of convolutions turns out to be cache inefficient [16]. Reversing the order, i.e. performing these as convolutions of matrix-vector products, yields a reduction of computation time of one order of magnitude and is therefore strongly recommended.

VII. APPLICATION

The application, considered here, is the FE model of an inductor featuring a ferromagnetic core (Fig. 3). The substantial saturation in the corner of the laminates introduces additional harmonics to the magnetic field. The deteriorated waveforms of voltages and currents injected by power electronic equipment, also contribute to the harmonic pollution in the inductor core. The non-linearity of the ferromagnetic core causes interference between all harmonic components which may lead to ferroresonant and chaotic behaviour of the electromagnetic system [17].

The accurate simulation of the steady-state operation of the inductor, is indispensable to detect local hot spots due

to additional losses, and the ageing of insulation material due to spikes in the voltages, during the design process. To this end, HBFEM is particularly attractive. The odd harmonics from the 1st up to the 7th are selected for the simulation. The initial FE mesh and the FE mesh after adaptive mesh refinement are shown in Fig. 4. The magnetic flux density is represented by its harmonic components in Fig. 5 and Fig. 6. Notice that the higher harmonic field plots are non-physical, i.e. they pretend non-existing currents in the magnetic core. This way of representating is, however, support the technical judgement as it clearly marks the places with significant saturation. Due to the numerical singularity at corner points, very high values for the magnetic flux density, i.e. 3.12 T, are observed (Fig. 6). Hence, the a-posteriori error estimator will always mark the FE elements near the corner for refinement. This infinite recursion is prevented by imposing a minimal size to the FE elements, i.e. smaller elements are excluded from further refinement. The magnetic flux densities at four points at the corner of the inductor core are extracted and plotted both in the time and the frequency domain (Fig. 7 and Fig. 8). The same model is also simulated by a transient FE solver. The results for steady-state operation are identical. The required computation time, however, is 2 hours and 10 minutes in the case of transient simulation and 20 minutes in the case of HBFEM, clearly indicating the benefits of HBFEM over the time stepping approach.

VIII. CONCLUSIONS

The harmonic balanced finite element method is implemented within an existing FE package relying upon overloading techniques. Especially for the quasi-static simulation of the steady-state operation of electrical energy apparatus, harmonic balanced simulation is advantageous over

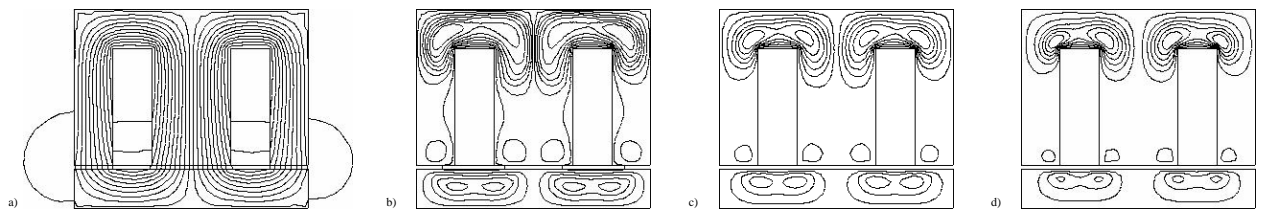


Fig. 5. (a-d) 1st, 3rd, 5th and 7th harmonic component of the magnetic flux lines in the inductor.

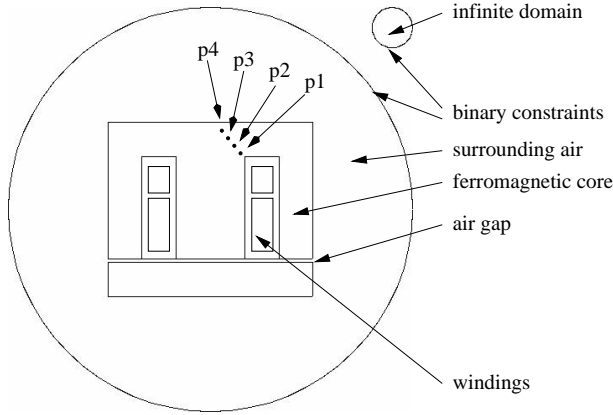


Fig. 3. Geometry of an inductor with ferromagnetic core.

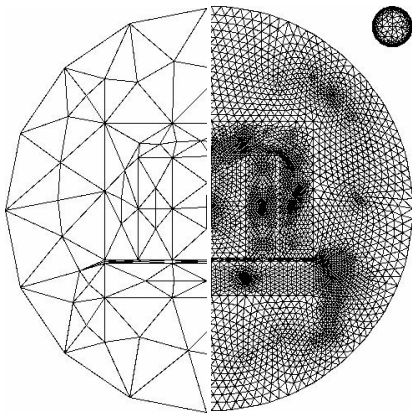


Fig. 4. Initial mesh and mesh after adaptive mesh refinement of the inductor model.

transient simulation. The selection of a finite and discrete harmonic spectrum to which all quantities of the simulation are restricted, relies upon technical considerations and allows the designer to focus upon the relevant phenomena in the device. The approach presented here, offers particular benefits over weak coupled or real equivalent variants, mainly because the corresponding system of equations is more feasible to be solved by iterative solvers. The application of the harmonic balanced finite element method to simulate the saturation and hence, the excitation of higher harmonics fields, in a ferromagnetic inductor core, motivates the use of harmonic balanced finite element simulation in electrotechnical design.

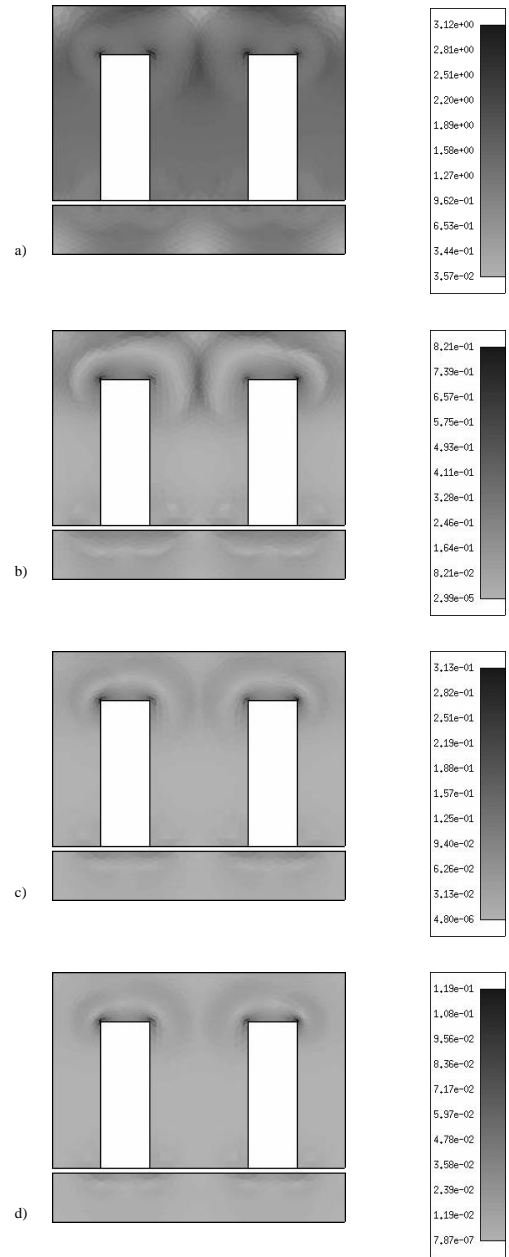


Fig. 6. (a-d) 1st, 3rd, 5th and 7th harmonic component of the magnetic flux density in the inductor core.

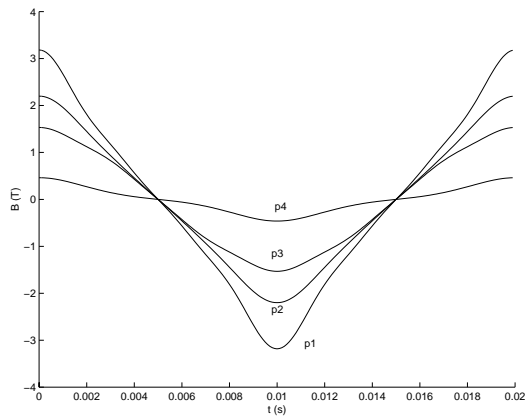


Fig. 7. Magnetic flux density variation in time at a point in the corner of the ferromagnetic core of the inductor.

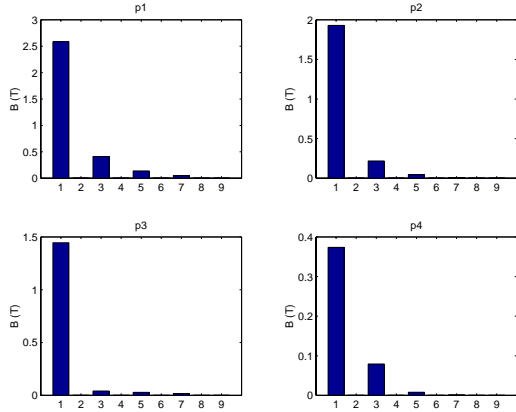


Fig. 8. Spectrum of the magnetic flux densities at four points in the inductor core as indicated in Fig. 3.

ACKNOWLEDGMENTS

The authors are grateful to the Belgian Fonds voor Wetenschappelijk Onderzoek Vlaanderen (G.0427.98) and the Belgian Ministry of Scientific Research (IUAP No. P4/20) for the financial support of this work.

REFERENCES

- [1] K. Hameyer and R. Belmans, *Numerical Modelling and Design of Electrical Machines and Devices*, WIT Press, Southampton, UK, 1999.
- [2] I.A. Tsukerman, A. Konrad, G. Bedrosian, and M.V.K. Chari, "A survey of numerical methods for transient eddy current problems," *IEEE Transactions on Magnetics*, vol. 29, no. 2, pp. 1711–1716, Mar. 1993.
- [3] E. Vassent, G. Meunier, and J.C. Sabonnadière, "Simulation of induction machine operation using complex magnetodynamic finite elements," *IEEE Transactions on Magnetics*, vol. 25, no. 4, pp. 3064–3066, July 1989.
- [4] S. Yamada, K. Bessho, and J. Lu, "Harmonic balance finite element method applied to nonlinear AC magnetic analysis," *IEEE Transactions on Magnetics*, vol. 25, no. 4, pp. 2971–2973, July 1989.
- [5] K. Eriksson, D. Estep, P. Hansbo, and C. Johnson, *Computational Differential Equations*, Cambridge University Press, Cambridge, 1996.
- [6] S. Yamada, P.P. Biringer, K. Hirano, and K. Bessho, "Finite element analysis of nonlinear dynamic magnetic field with DC component in the harmonic domain," *IEEE Transactions on Magnetics*, vol. 26, no. 5, pp. 2199–2201, Sept. 1990.

- [7] J. Driesen and K. Hameyer, "The simulation of magnetic problems with combined fast and slow dynamics using a transient time-harmonic method," submitted to *The European Physical Journal*, 2000.
- [8] H. De Gersem, S. Vandewalle, and K. Hameyer, "Krylov subspace methods for harmonic balanced finite element methods," in *Proceedings of the 3rd International Workshop on Scientific Computing in Electrical Engineering (SCEE-2000)*, Warnemünde, Germany, Aug. 2000, p. 52.
- [9] Y. Saad, *Iterative Methods for Sparse Linear Systems*, PWS Publishing Company, Boston, 1996.
- [10] R.W. Freund, "Conjugate gradient-type methods for linear systems with complex symmetric coefficient matrices," *SIAM Journal on Scientific Computing*, vol. 13, pp. 425–448, Jan. 1992.
- [11] D. Lederer, H. Igarashi, and A. Kost, "The Newton-Raphson method for complex equation systems," in *Proceedings of the 7th international IGTE symposium on numerical field calculation in electrical engineering (IGTE98)*, Graz, Austria, Sept. 1996, pp. 391–394.
- [12] U. Pahner, R. Mertens, H. De Gersem, R. Belmans, and K. Hameyer, "A parametric finite element environment tuned for numerical optimization," *IEEE Transactions on Magnetics*, vol. 35, no. 5, pp. 2936–2939, Sept. 1998.
- [13] G. Booch, *Object-Oriented Analysis and Design*, The Benjamin/Cummings Publishing Company, Redwood City, 2nd edition, 1994.
- [14] B. Stroustrup, *The C++ Programming Language*, Addison-Wesley Publishing Company, Reading, Massachusetts, USA, 1991.
- [15] G.H. Golub and C.F. Van Loan, *Matrix Computations*, The John Hopkins University Press, Baltimore, 1989.
- [16] H. De Gersem, S. Vandewalle, and K. Hameyer, "Krylov subspace methods for harmonic balanced finite element methods," accepted for publication in *Lecture Notes in Computational Science and Engineering*, 2001.
- [17] T. Van Craenenbroeck, W. Michiels, D. Van Dommelen, and K. Lust, "Bifurcation analysis of three-phase ferroresonant oscillations in ungrounded power systems," *IEEE Transactions on Power Delivery*, vol. 14, no. 2, pp. 531–536, Apr. 1999.