Convergence Improvement of the Conjugate Gradient Iterative Method for Finite Element Simulations

H. De Gersem and K. Hameyer

Katholieke Universiteit Leuven, Dep. EE (ESAT) / Div. ELEN
Kardinaal Mercierlaan 94, B-3001 Leuven, Belgium
tel.: +32 16 32 10 20 – fax: +32 16 32 19 85 – e-mail: Herbert.DeGersem@esat.kuleuven.ac.be

Abstract —The slow convergence of the Incomplete Cholesky preconditioned Conjugate Gradient (CG) method, applied to solve the system representing a magnetostatic finite element model, is caused by the presence of a few little eigenvalues in the spectrum of the system matrix. The corresponding eigenvectors reflect large relative differences in permeability. A significant convergence improvement is achieved by supplying vectors that span approximately the partial eigenspace formed by the slowly converging eigenmodes, to a deflated version of the CG algorithm. The numerical experiments show that even roughly determined eigenvectors already bring a significant convergence improvement. The deflating technique is embedded in the simulation procedure for a permanent magnet DC machine.

Keywords —Deflation, finite element method, iterative methods, Krylov subspace, spectrum, electrical machines.

Introduction

Finite element simulation is embedded in design and optimisation procedures for electromagnetic devices. Finite element simulation commonly provides the ability to consider arbitrary geometries, nonlinear materials and eddy currents effects. An important drawback compared to smaller models such as magnetic equivalent circuits and models coming from point mirroring techniques, are the huge computational expenses, especially if 3D simulation is required.

As the finite element calculation is usually part of a wider electromagnetic simulation procedure, often smaller models of the same devices already exist. The finite element model is, however, built and solved starting from scratch. In this paper, the availability of a small-sized alternative modellisation is exploited to enhance the convergence of the iterative solver within the finite element simulation software.

Magnetostatic model

A discrete 2D magnetostatic model is represented by the system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{b} . \tag{1}$$

Here, $\mathbf{A}$ and $\mathbf{b}$ result from discretising the magnetostatic partial differential equation

$$-\nabla \cdot \left(\nu \nabla A_z\right) = J_z \tag{2}$$

with A_z and J_z the *z*-components of the magnetic vector potential and the current density and ν the reluctivity, by e.g. linear triangular finite elements (Silvester, 1990). $\mathbf{x}$ is the vector of the nodal magnetic vector potentials. $\mathbf{A}$ is symmetric and positive definite and n denotes its dimension.

Convergence of the iterative solution

As the system matrix is sparse and the spectral properties of the system matrix are well-known, Krylov subspace iterative methods are appropriate (Saad, 1996). The Conjugate Gradient (CG) method is suited for symmetric, positive definite systems. These methods search for the solution within the Krylov subspace, which dimension is augmented by one in each iteration step. Theoretically the exact solution is reached after n iteration steps. However, it is expected to reach an acceptable accuracy in much fewer steps. The convergence history of CG applied to an example model is plotted in Fig. 1. The error is bound by

$$\left\|\mathbf{x} - \mathbf{x}^{(k)}\right\|_{\mathbf{A}} \leq 2\left\|\mathbf{x} - \mathbf{x}^{(0)}\right\|_{\mathbf{A}} \left(\frac{\sqrt{K} - 1}{\sqrt{K} + 1}\right)^k , \tag{3}$$

with $\mathbf{x}$ the exact solution and $\mathbf{x}^{(k)}$ the approximative solution at iteration step k . The condition number K is

$$K = \frac{\lambda_{\max}}{\lambda_{\min}} , \tag{4}$$

with $\lambda_{\max}$ the largest and $\lambda_{\min}$ the smallest eigenvalue of $\mathbf{A}$. Therefore, it is possible to obtain information about the convergence of Krylov subspace solvers by interpreting the spectrum of the system matrix (Fig. 2).

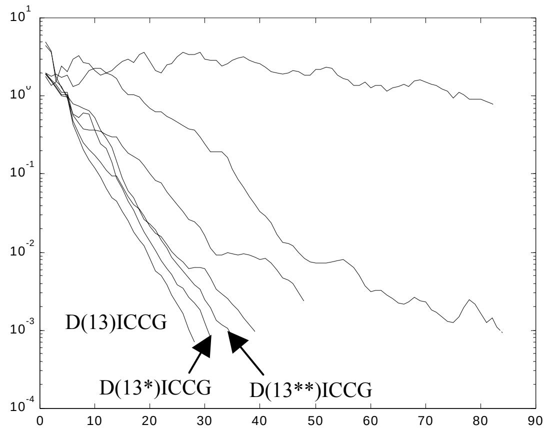

Fig. 1: Convergence histories of ICCG, D(3)ICCG, D(6)ICCG, D(13)ICCG, D(13*)ICCG and D(13**)ICCG.

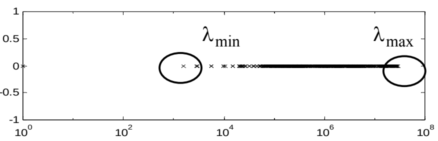

Fig. 2: Spectrum of $\mathbf{A}$.

Preconditioning

The convergence of a Krylov subspace iterative method can substantially be enhanced by applying an appropriate preconditioner $\mathbf{M}$ to the system (Saad, 1996):

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{x} = \mathbf{M}^{-1}\mathbf{b} \, . \tag{5}$$

Here, as a preconditioner, an Incomplete Cholesky (IC) factorisation is applied. The IC factor $\mathbf{L}$ is computed in advance keeping the same sparsity pattern as $\mathbf{A}$. The preconditioner is $\mathbf{M} = \mathbf{L}\mathbf{L}^{\mathrm{T}}$. The convergence and the spectrum of the preconditioned system are are denoted by ICCG in Fig. 1 and Fig. 3 respectively. The eigenvalues of the preconditioned system are much more clustered when compared to those of the original system. The outlayers are also located more at the center of the spectrum. Therefore, the condition number improved (Table I). Some outlayers still remain isolated from the rest of the spectrum. The presence of these few eigenmodes have a harmful influence on the condition number and thus the convergence of CG. If these eigenmodes can be removed from the system spectrum, a far better convergence is expected. Therefore, a closer look at these eigenmodes is motivated.

The origin of these slowly converging eigenmodes is to be found in the properties of the differential model itself. The flux patterns corresponding to the eigenvectors are shown in Fig. 4. They reflect long range effects and large relative differences in permeability present in the model. The eigenvectors represent the magnetic fields formed by an independent set of possible rotor excitations. In Fig. 3a, it is seen that about 13 eigenvalues are isolated from the rest of the spectrum. As a consequence, it is plausible to ascribe the slowly convergence to the slotting of the motor.

TABLE I: CONDITION NUMBERS AND ITERATION COUNTS.

Solver	System Matrix	Condition Number	Number of Iterations
CG	$\mathbf{A}$	2.92e7	264
ICCG	$\mathbf{M}^{-1}\mathbf{A}$	870	84
D(3)ICCG	$\mathbf{M}^{-1}\mathbf{P}_3^{\mathrm{T}}\mathbf{A}$	308	52
D(6)ICCG	$\mathbf{M}^{-1}\mathbf{P}_6^{\mathrm{T}}\mathbf{A}$	135	39
D(13)ICCG	$\mathbf{M}^{-1}\mathbf{P}_{13}^{\mathrm{T}}\mathbf{A}$	51.2	28
D(13*)ICCG	$\mathbf{M}^{-1}\mathbf{P}_{13*}^{\mathrm{T}}\mathbf{A}$	66.9	31
D(13**)ICCG	$\mathbf{M}^{-1}\mathbf{P}_{13**}^{\mathrm{T}}\mathbf{A}$	73.9	35

(a)
(b)
(c)
(d)
(e)

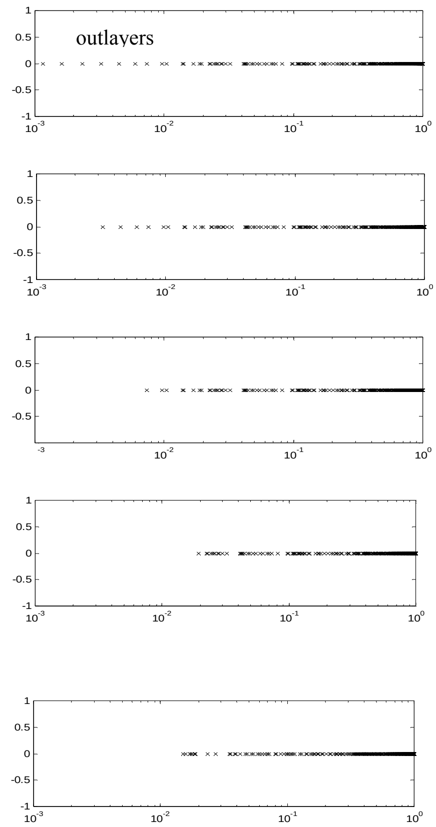

Fig. 3: Spectra of the systems corresponding to (a) ICCG, (b) D(3)ICCG, (c) D(6)ICCG, (d) D(13)ICCG and (e) D(13*)ICCG.

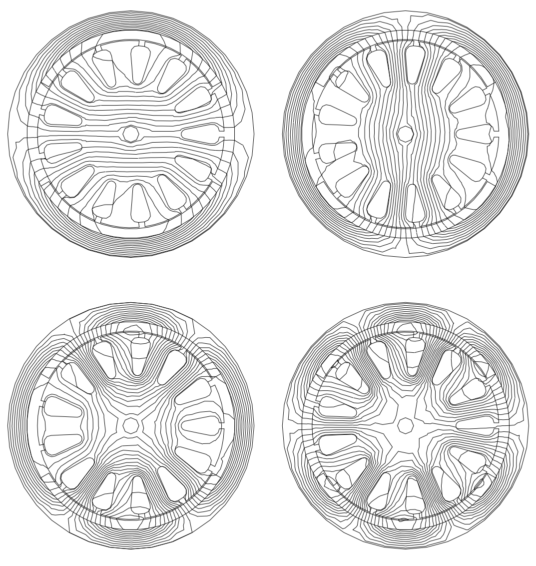

Fig. 4: Flux patterns corresponding to the eigenvectors related to the four smallest eigenvalues of $\mathbf{M}^{-1}\mathbf{A}$.

Exact Deflation

To annihilate the effect of *m* slowly converging eigenmodes, the *m* corresponding eigenvectors are removed out of the space in which the solution is searched. If the columns of a matrix $\mathbf{V}$ span an approximative eigenspace V , the projector

$$\mathbf{P} = \mathbf{I} - \mathbf{V}\mathbf{E}^{-1}(\mathbf{A}\mathbf{V})^{\mathrm{T}} \tag{6}$$

with $\mathbf{E} = (\mathbf{A}\mathbf{V})^{\mathrm{T}}\mathbf{V}$ projects a vector $\mathbf{u}$ to the vector $\mathbf{Pu}$ into $V^{\perp}$, the space orthogonal to V (Vuik, 1999). Applying $\mathbf{I}-\mathbf{P}$ to $\mathbf{u}$ yields the vector $(\mathbf{I}-\mathbf{P})\mathbf{u}$ contained in V .

The projector defines a decomposition of the *n*-dimensional search space into the *m*-dimensional partial eigenspace V and the $(n-m)$-dimensional deflated eigenspace $V^{\perp}$. The solution of (1) consists of a part $\mathbf{x}_1 \in V$ and a part $\mathbf{x}_2 \in V^{\perp}$. The part contained in the partial eigenspace is computed as

$$\mathbf{x}_1 = \mathbf{V}\mathbf{E}^{-1}\mathbf{V}^{\mathrm{T}}\mathbf{b}\,. \tag{7}$$

This calculation is inexpensive because $\mathbf{E}$ is of dimension *m* and usually only a few eigenvectors are selected for deflation. The part orthogonal to V is solved from

$$\mathbf{M}^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{x}_2 = \mathbf{M}^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{b}\,. \tag{8}$$

System (8) is singular. However, the Krylov subspace solver is capable of solving such systems if the righthandside is contained within the range of the system matrix (Kaasschieter, 1988), as is here:

$$\mathbf{M}^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{b} \in \mathrm{Ran}\left(\mathbf{M}^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{A}\right). \tag{9}$$

The convergences of the preconditioned systems deflated by 3,6 and 13 eigenvectors are denoted by D(3)ICCG, D(6)ICCG and D(13)ICCG respectively (Fig. 1). The spectra are collected in Fig. 3. The slowly converging eigenvalues are projected upon zero (not visible in Fig. 3 because of the logarithmic axis) and indicate the singularity of the deflated systems. The improvement of the condition number and the convergence is also clear from Table I.

Approximative Deflation

The previous section assumes that the eigenvectors related to the small converging modes are available. Determining one eigenvector, however, is as expensive as solving the original linear system. As a consequence, the approach presented until now, is not advantageous.

As already mentioned, the eigenvectors have typical shapes corresponding to the flux patterns that are commonly sketched by the design engineer intuitively. It is possible to construct, based on geometrical reasoning, a few base vectors of eigenspace approximating the exact partial eigenspace V . The determination procedure for these vectors is easily embedded within software computing the magnetic reluctance required for a magnetic equivalent circuit model. The crucial assumption here is the fact that these vectors not only do approximate the eigenvectors of $\mathbf{A}$ but also those of $\mathbf{M}^{-1}\mathbf{A}$. This resemblance is implicitly verified by the numerical experiments below.

The convergence of the approximately deflated system, denoted by D(13*)ICCG and its spectrum are plotted in Fig. 1 and Fig. 3 respectively. The numerical experiments indicate that even a rough determination of the partial eigenvector space suffices to enhance the convergence of the finite element solution substantially.

Application

The deflated solver is applied to simulate a permanent magnet DC motor. Two radially magnetised permanent magnets excite a magnetostatic field in the air gap of the device. DC currents through the windings of the rotor cause armature reaction (Fig. 5).

The design of the device can be based on semi-analytical formulae. Correction factors, e.g. the Carter factor dealing with the slotting of the machine, are computed relying upon finite element models whereas the overall behaviour of the device is simulated by analytic expressions. While computing the Carter factor, the same reduced model of one slot pitch is used to compute a local flux associated with the excitation of one rotor slot (Fig. 6). This pattern is mirrored and rotated to obtain 13 independent base vectors spanning a space that is approximating the slowly converging eigenspace of $\mathbf{M}^{-1}\mathbf{A}$ (Fig. 7). The local support of these base vectors enable an efficient application of the projector within the CG algorithm. The same projector is used to deflate the systems associated with the different Newton steps dealing with the nonlinear material characteristics.

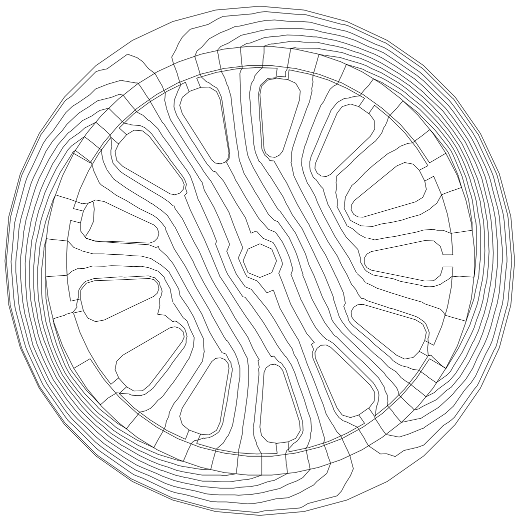

Fig. 5: Magnetic flux line plot of a permanent magnet DC machine.

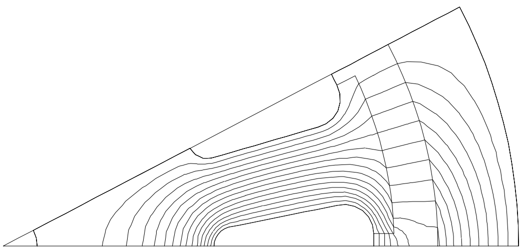

Fig. 6: Reduced finite element model of one slot.

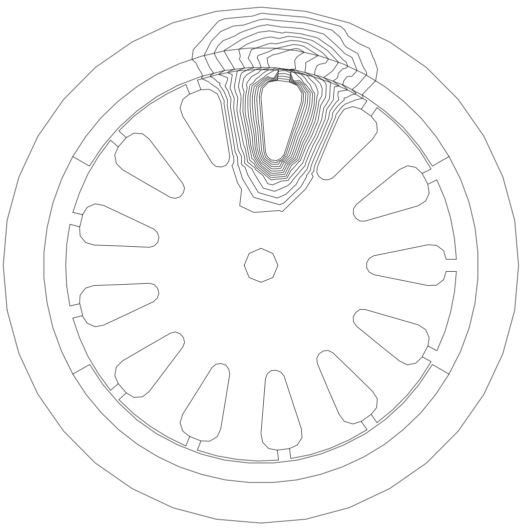

Fig. 7: Magnetic flux lines corresponding to a base vector of an approximative eigenspace of the system matrix.

13 eigenvectors computed by a geometrical algorithm determining flux paths through electromagnetic devices and computing magnetic reluctances, define a more rough projection $\mathbf{P}_{13**}$. Also for this deflation, the convergence enhancement is substantial (Fig. 1 and Table I).

Conclusions

After standard preconditioning, a few relatively small eigenvalues remain in the spectrum of the system matrix, causing slow convergence of the Conjugate Gradient iterative method. Defining a projector subtracting the subspace spanned by the corresponding eigenvectors from the Krylov subspace, results in a deflated algorithm with improved convergence properties. The eigenvectors have a physical meaning. Using an alternative rough modellisation technique, it is possible to construct a base for a space approximating the partial eigenspace. The approximately deflated Conjugate Gradient algorithm still establishes a good convergence.

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