Convergence Improvement of the Conjugate Gradient Iterative Method for Finite Element Simulations

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Abstract —The slow convergence of the Incomplete Cholesky preconditioned Conjugate Gradient (CG) method, applied to solve the system representing a magnetostatic finite element model, is caused by the presence of a few little eigenvalues in the spectrum of the system matrix. The corresponding eigenvectors reflect large relative differences in permeability. A significant convergence improvement is achieved by supplying vectors that span approximately the partial eigenspace formed by the slowly converging eigenmodes, to a deflated version of the CG algorithm.

1. Introduction

An important drawback of finite element models are the huge computational expenses. As the finite element calculation is usually part of a wider electromagnetic simulation procedure, often smaller models of the same devices already exists. In this paper, an available small-sized alternative model is exploited to enhance the convergence of the iterative solver within the finite element simulation.

A discrete 2D magnetostatic model is represented by the system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}\,. \tag{1}$$

Here, $\mathbf{A}$ and $\mathbf{b}$ result from discretising a magnetostatic partial differential equation [1]. $\mathbf{x}$ is the vector of the nodal magnetic vector potentials. $\mathbf{A}$ is symmetric and positive definite and n denotes its dimension.

2. Convergence of the Conjugate Gradient Method

The Conjugate Gradient (CG) method is a Krylov subspace iterative method appropriate to solve large, sparse, symmetric, positive definite systems [2]. The convergence history of CG applied to an example model is plotted in Fig. 1. The error is bound by

$$\left\|\mathbf{x}-\mathbf{x}^{(k)}\right\|_{\mathbf{A}} \le 2\left\|\mathbf{x}-\mathbf{x}^{(0)}\right\|_{\mathbf{A}}\left(\frac{\sqrt{K}-1}{\sqrt{K}+1}\right)^{k}, \tag{2}$$

with $\mathbf{x}$ the exact solution and $\mathbf{x}^{(k)}$ the approximative solution at iteration step k. The condition number K is the ratio between the largest and the smallest eigenvalue of $\mathbf{A}$. Therefore, it is possible to obtain information about the convergence of Krylov subspace solvers by interpreting the spectrum of the system matrix (Fig. 2).

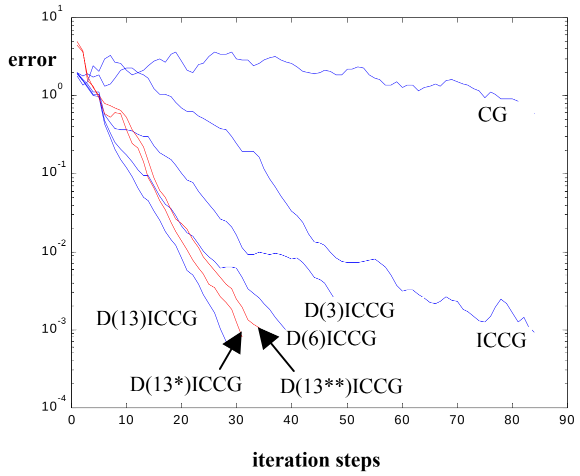

Fig. 1: Convergence histories of CG, ICCG, D(3)ICCG, D(6)ICCG, D(13)ICCG, D(13*)ICCG and D(13**)ICCG.

3. Preconditioning

The convergence of CG can substantially be enhanced by applying an Incomplete Cholesky (IC) preconditioner $\mathbf{M}$ to the system [2].

$$\mathbf{M}^{-1}\mathbf{A}\mathbf{x} = \mathbf{M}^{-1}\mathbf{b}\,. \tag{3}$$

The eigenvalues of the preconditioned system are much more clustered when compared to those of the original system, yielding a better condition number (Table I). The few remaining outlayers in the spectrum, however, have a harmful influence on K and thus on the convergence of CG (Fig. 2). The origin of these slowly converging eigenmodes can be found in the properties of the differential model itself. The flux patterns of the corresponding eigenvectors reflect the long range effects and the large relative differences in permeability present in the model (Fig. 3).

TABLE I: CONDITION NUMBERS AND ITERATION COUNTS.

Solver	System Matrix	Condition Number	Number of Iterations
CG	$\mathbf{A}$	2.92e7	264
ICCG	$\mathbf{M}^{-1}\mathbf{A}$	870	84
D(3)ICCG	$\mathbf{M}^{-1}\mathbf{P}_3^{\mathrm{T}}\mathbf{A}$	308	52
D(6)ICCG	$\mathbf{M}^{-1}\mathbf{P}_6^{\mathrm{T}}\mathbf{A}$	135	39
D(13)ICCG	$\mathbf{M}^{-1}\mathbf{P}_{13}^{\mathrm{T}}\mathbf{A}$	51.2	28
D(13*)ICCG	$\mathbf{M}^{-1}\mathbf{P}_{13*}^{\mathrm{T}}\mathbf{A}$	66.9	31
D(13**)ICCG	$\mathbf{M}^{-1}\mathbf{P}_{13**}^{\mathrm{T}}\mathbf{A}$	73.9	35

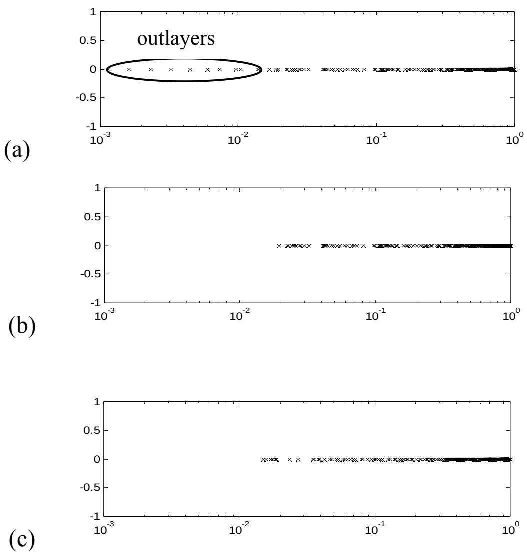

Fig. 2: Spectra of the systems corresponding to (a) ICCG, (b) D(13)ICCG and (c) D(13*)ICCG.

4. Deflated ICCG

It is possible to annihilate the effect of slowly converging eigenmodes. If the *m* columns of a matrix $\mathbf{V}$ span a partial eigenspace V, the projector

$$\mathbf{P} = \mathbf{I} - \mathbf{V}\mathbf{E}^{-1}(\mathbf{A}\mathbf{V})^{\mathrm{T}} \tag{4}$$

with $\mathbf{E} = (\mathbf{A}\mathbf{V})^{\mathrm{T}}\mathbf{V}$ projects a vector onto $V^{\perp}$, the space orthogonal to V [3]. The solution of (1) is split up into a part $\mathbf{x}_1 \in V$ and a part $\mathbf{x}_2 \in V^{\perp}$. Computing

$$\mathbf{x}_1 = \mathbf{V}\mathbf{E}^{-1}\mathbf{V}^{\mathrm{T}}\mathbf{b}\,. \tag{5}$$

is inexpensive because $\mathbf{E}$ is of dimension *m* and usually only a few eigenvectors are selected for deflation. $\mathbf{x}_2$ is solved from

$$\mathbf{M}^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{A}\mathbf{x}_2 = \mathbf{M}^{-1}\mathbf{P}^{\mathrm{T}}\mathbf{b}\,. \tag{6}$$

System (6) is singular. However, CG is capable of solving such systems if the righthandside is contained within the range of the system matrix [4], as is here. The improvement of the condition number and the convergence of the deflated system (denoted by D(*m*)ICCG) is clear from Fig. 1, Fig. 2 and Table I.

As determining one eigenvector is as expensive as solving the original linear system, the approach presented until now is not advantageous. It is however possible to construct, based on geometrical reasoning, a few base vectors of a space approximating V. The convergence and the spectrum of an approximately deflated systems, denoted by D(13*)ICCG, is shown in Fig. 1. The numerical experiments indicate that even a rough determination of V suffices to enhance the convergence of the finite element solution substantially (D(13**)ICCG in Table I).

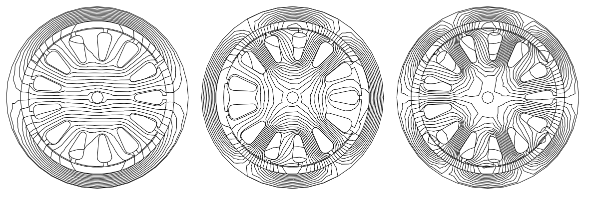

Fig. 3: Flux patterns corresponding to the eigenvectors related to three small eigenvalues of $\mathbf{M}^{-1}\mathbf{A}$.

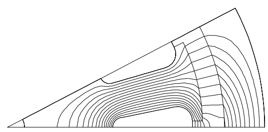

Fig. 4: Reduced finite element model of one slot.

5. Application

The deflated solver is applied to simulate a permanent magnet DC motor. The slow convergence of CG applied to the motor model is related to the slotting of the rotor. A reduced model of one slot pitch is used to compute a local flux associated with the excitation of one rotor slot (Fig. 4). This pattern is mirrored and rotated to obtain 13 independent base vectors related to the 13 rotor slots and spanning an approximative eigenspace of $\mathbf{M}^{-1}\mathbf{A}$. The local support of these base vectors enable an efficient application of the projector within the CG algorithm. The convergence enhancement is substantial (D(13*)ICCG in Fig. 1).

6. Conclusions

A few relatively small eigenvalues cause slow convergence of the preconditioned Conjugate Gradient method. Defining a projector subtracting a subspace spanned by the corresponding eigenvectors from the Krylov subspace, results in a deflated algorithm with improved convergence.

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