Adaptive Mesh Generation Methods for Coupled Field Problems

Abstract — This paper discusses methods to produce adaptively refined finite element meshes, to be used in coupled field problems. Error estimation information from one subproblem mesh is transferred using a projection methods to the other meshes and combined, by means of different types of mathematical averaging, with the other error vectors before refinement is performed. In this way, mesh compatibility is assured to maintain the required accuracy in the coupled problem. This approach is applicable to multi-physics problems such as coupled electromagnetic-thermal problems involving eddy current losses and multi-harmonic problems of the same physical nature.

Keywords — Coupled electromagnetic-thermal problems, finite elements, mesh refinement

1. Introduction

Coupled problems simulated using the finite element method (FEM), are best solved on individual meshes (Eustache, 1996), since the physical reality of the subproblem imposes different boundary conditions and accuracy requirements and since the subproblem domains often only have a part of the model in common. These meshes are usually constructed in advance. Occasionally, they are adaptively refined on an individual basis, but thereby neglecting the associated coupled problem. It can be necessary to locally apply different types of finite elements in the individual subproblem.

The use of entirely identical meshes is therefore not a good idea. Identical meshes mean wasting elements and therefore an increase of the computational efforts. For example, if one subproblem has a rather smooth solution and the sensitivities of the other subproblems with respect to the solution in this region are relatively low, the use of a fine mesh for this subproblem is not necessary. Sometimes a mesh is constructed locally too fine for numerical reasons e.g. to obtain high quality elements in the transition between coarsely and fine meshed regions for one subproblem. If these requirements are not relevant for other subproblems, then this mesh does not have to be as fine as the other. It is possible that strategies to enhance the geometric mesh quality yield slightly different local meshes.

Assuming no singularities are present in the vicinity of the element, the order of the error for the FEM model can be expressed as a proportionality in terms of a relevant characteristic element size h and the polynomial order of the element p by (Zienkiewicz, 1994):

$$e \sim O\left(h^{p+1}\right) \tag{1}$$

Overlapping mesh parts with significantly different element size should be avoided or the element approximation must be of an appropriate polynomial order. In fact, the overall accuracy of the coupled solution is determined by the lowest accuracy of the individual subproblems considered. When the polynomial order of the elements is locally the same, the sizes may not differ significantly. This property, which can be called ‘mesh compatibility’, has to be kept, even after mesh refinement. To prevent the generation of incompatibilities, error information has to be transferred from one mesh onto the other during the adaptive mesh generation process.

2. Error information transfer

A. Method principle

When a subproblem-specific error estimator indicates that it is required to refine the mesh in a particular location, this information needs to be transferred to the mesh of the other subproblem and combined with the locally estimated error. This must be performed in such a way, that the local normalised error will increase when refinement is strictly required in the other associated subproblem. Mathematically, this error can be interpreted as a field quantity expressed per element. For the total mesh of subproblem i, the vector of local element errors $\{e_i\}$ is obtained and is normalised.

Fig. 1 illustrates a methodology (for two coupled subproblems) to maintain a higher degree of mesh compatibility. The error information vectors of the individual problems are projected to the other subproblem mesh. Hence, an approximate local error estimation for all the involved subproblems is available in every submesh element to be combined to form a global estimate.

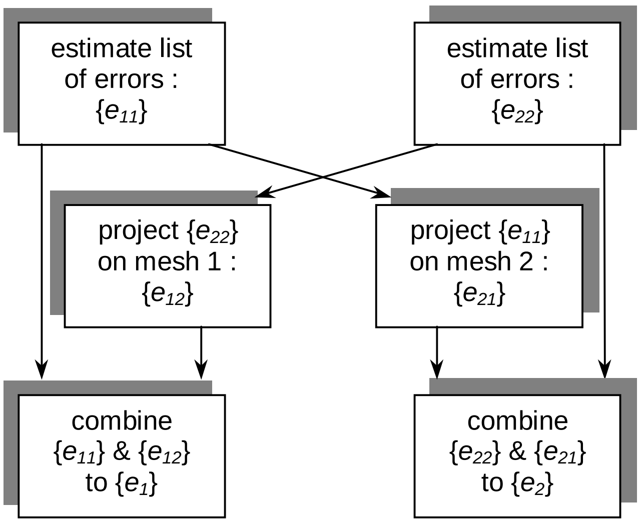


Fig. 1. Methodology to maintain mesh compatibility in coupled field error estimation.

B. Error combination

The 'combination' of the error vectors is obtained by averaging the normalised errors. This averaging, yielding a new set of normalised vectors, may be (for *n* subfields):

1) algebraic:

$$e_i = \frac{1}{n}\sum_j \left|e_{ij}\right| \qquad (2)$$

$$e_i = \frac{\sum_j w_{ij}\left|e_{ij}\right|}{\sum_j w_{ij}} \qquad (3)$$

2) quadratic:

$$e_i = \sqrt{\frac{\sum_j \left|e_{ij}\right|^2}{n}} \qquad (4)$$

$$e_i = \sqrt{\frac{\sum_j w_{ij}\left|e_{ij}\right|^2}{\sum_j w_{ij}}} \qquad (5)$$

3) geometric:

$$e_i = \left(\prod_j \left|e_{ij}\right|\right)^{\frac{1}{n}} \qquad (6)$$

$$e_i = \left(\prod_j \left|e_{ij}\right|^{w_{ij}}\right)^{\frac{1}{\sum_j w_{ij}}} \qquad (7)$$

In the weighted averages, the weight w_{ii} can be fixed to be dominant. The quadratic mean stresses extreme values: if the error is significant in one subfield, this is retained in all fields. This expression can guarantee a high degree of 'mesh compatibility'. To a lower extent, the algebraic mean has the same property. The geometric mean smoothes the element errors: the element only gets a large error estimate when it has significant errors in all subfields. A high combined value may therefore indicate a locally strong mutual dependence if appropriate error estimators for the individual problems were selected.

C. Projection implementation

The projection operation for the error quantities, interpreted as a piece-wise continuous low-order field in case of zero-order estimates can be implemented in two ways:

1) *Interpolation:* An average over the destination element's area is calculated using numerical integration by means of appropriate Gauss points (this technique can be used to transfer losses and other low-order fields (Driesen, 1998).

$$e(\Omega_e) = \frac{1}{\Omega_e}\sum_{k=1}^{n_{i_1}}\left(e_{i_1 k}\int_{\Omega_e}\left(N_k^{i_1}\right)d\Omega\right) \qquad (8)$$

2) *Least-squares or weighted residual method:* An alternative way to transfer the error information from one mesh, associated to subproblem i_1, onto another mesh, associated with subproblem i_2, is a least-squares fit. The quadratic difference error field, written as a weighted sum of shape functions $N_k^{i_1}$ and $N_k^{i_2}$, is minimised for every unknown error term $e_{i_2 k}$:

$$\frac{\partial}{\partial e_{i_2 k}}\int_{\Omega_e}\left(\sum_{k=1}^{n_{i_2}}\left(N_k^{i_2}e_{i_2 k}\right)-\sum_{k=1}^{n_{i_1}}\left(N_k^{i_1}e_{i_1 k}\right)\right)^2 d\Omega = 0 \qquad (9)$$

This yields:

$$\int_{\Omega_e}\left(N_k^{i_2}\left(\sum_{k=1}^{n_{i_2}}\left(N_k^{i_2}e_{i_2 k}\right)-\sum_{k=1}^{n_{i1}}\left(N_k^{i_1}e_{i_1 k}\right)\right)\right)d\Omega = 0 \qquad (10)$$

Eq. (10) can be interpreted as a weighted residual. The sparse system to be solved contains building blocks similar to those found in FEM systems.

$$\sum_{k=1}^{n_{i_2}}\left(\int_{\Omega_e}\left(N_l^{i_2}N_k^{i_2}\right)d\Omega\right)e_{i_2 k} = \sum_{k=1}^{n_{i_1}}\left(e_{i_1 k}\int_{\Omega_e}\left(N_l^{i_2}N_k^{i_1}\right)d\Omega\right) \qquad (11)$$

The integrals in the right-hand side expression are to be evaluated numerically as they contain partially overlapping shape functions (Fig. 2). In the case of zero order errors, the interpolation and least-squares approaches become identical. For higher order estimates, the sparse equation system has to be solved.

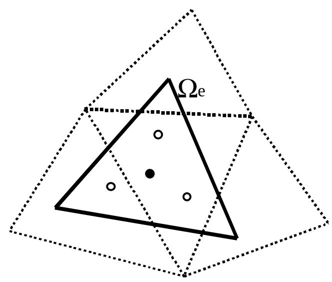


Fig. 2. Related mesh elements in a projection operation; some Gauss points for numerical integration are indicated.

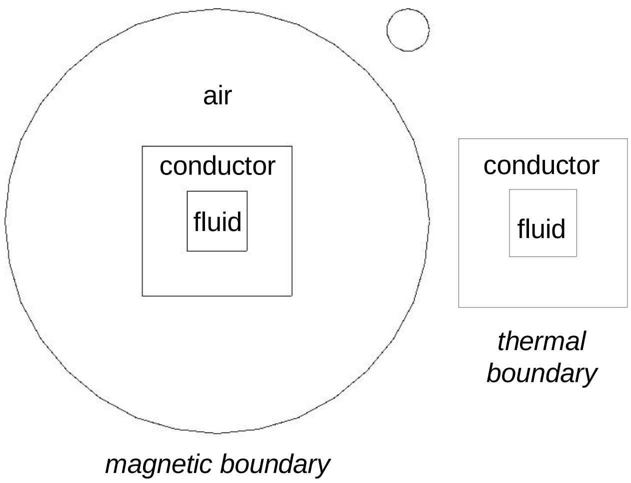
3 Applications

A. Electromagnetic-thermal problems

The presented adaptive mesh refinement technique is used in coupled electromagnetic-thermal problems. In this type of problems, different meshes have to be used because of the different physical properties of the coupled problem's subdomains. For instance, the air region is entirely discretised for the magnetic field as it carries the leakage flux. In general, this part is replaced by convective boundary conditions for the thermal problem definition.

The joint problem is often solved first on a set of meshes generated by an initial solution of the problems in an uncoupled way. This leads to a (simple) initial mesh with a sufficient quality to start up the coupled problem. In a next step, the problem is solved in a coupled way, with *h*-adaptive refinements. This coupled problem solution can be obtained by means of substitution or (quasi-)Newton algorithms (Eustache, 1996), (Driesen, 2000), (Molfino, 1989).

The example discussed here is a conductive heating problem. A square shaped tube constructed out of electrically conducting material heats a fluid flowing internally through the cooling channel. The surrounding air is included in the magnetic domain (Fig. 3a). In the thermal domain, convective boundary conditions are present internally and externally (Fig. 3b). Due to skin effect, the current and loss density is distributed over the cross-section. First order triangular elements are used for the field solution. The non-linear material characteristics and loss densities are assumed to be uniform within a finite element (zero order approximation).



(a) magnetic domain (b) thermal domain

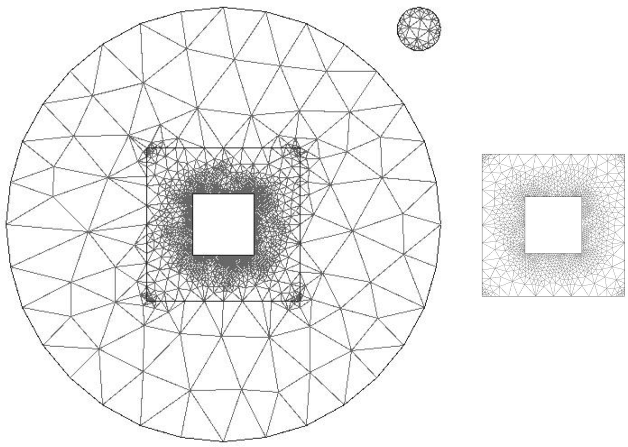
Fig. 3. Coupled problem domains.

The error estimator used for the thermal problem is based on the difference of the thermal gradient or the heat flux through an element as an indication for possible temperature differences. An adaptive refinement of the thermal problem by itself would achieve a mesh that is mostly refined in the vicinity of the cooling channel.

The magnetic field error estimator is chosen to yield an accurate loss density distribution. Therefore, the difference between the current densities is used for the error estimation. As skin effect is considered, a fine mesh in the conductor part close to the surface is obtained, which is significantly different from the mesh defining the thermal field problem.

This is illustrated in Figures 4 and 5. In the refinement strategies used, the projection is implemented as interpolation.

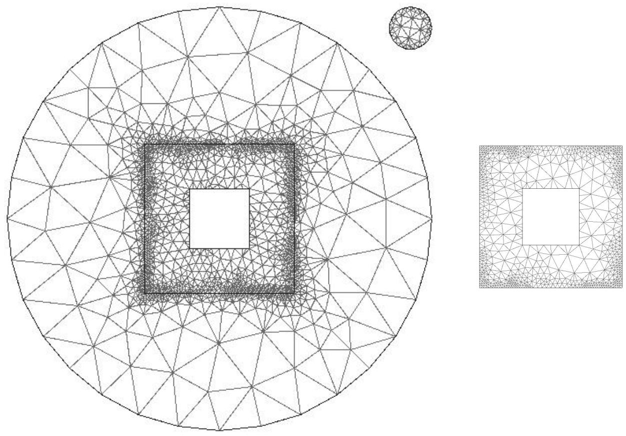
When the thermal error estimation is dominant (e.g. a large weighting coefficient in (3)) the meshes of Fig. 4 are obtained. The magnetic field is refined close to the channel due to the thermal error transfer. The skin region is only refined in the corners, where the (non-dominant) magnetic error estimates are large in magnitude.



(a) magnetic field mesh (b) thermal field mesh

Fig. 4. Meshes obtained using adaptive refinement with a large weight for the thermal field error estimation.

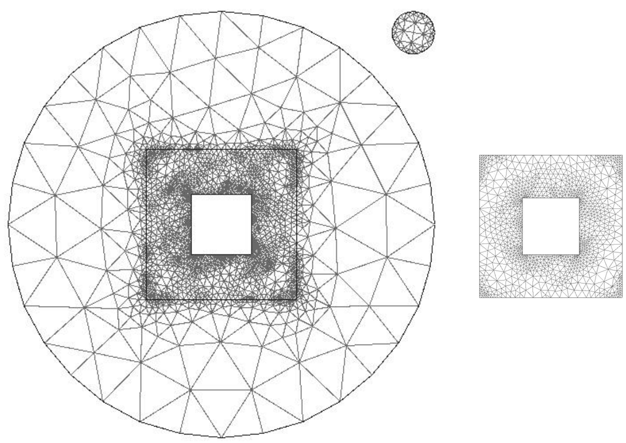
On the other hand, when the magnetic error estimation dominates, the skin regions will contain most of the elements in compatible thermal and magnetic meshes (Fig. 4).



(a) magnetic field mesh (b) thermal field mesh

Fig. 5. Meshes obtained using adaptive refinement with a large weight for the magnetic field error estimation.

To obtain an accurate coupled solution, it is important that the mesh in the conductive region stays compatible and be refined both, close to the skin and in the vicinity of the cooling channel. The result of the combined error estimation is shown in Fig. 6. To obtain these meshes, the quadratic averaging (4) is used, with equal weights for both subproblem error estimations.



(a) magnetic field mesh (b) thermal field mesh

Fig. 6. Meshes obtained using adaptive refinement with an equal weight for the magnetic and thermal field error estimation.

To obtain these meshes, four refinement steps were employed. In each step, the number of elements is approximately doubled by refining the elements with the largest scaled combined error. After the refinement, extra mesh enhancing operations such as local node movements are applied (Hameyer, 1999).

B. Multi-harmonic problems

Another application involves the coupling of different solution fields of the same physical nature. More in particular, in frequency domain methods using more than one harmonic (Driesen, 1999), a set of coupled eddy-current problems is to be solved. Since each frequency has a different skin depth, the meshes do not have to be identical. To achieve a sufficient accuracy of the solution and the post-processing (e.g. when the total current density is to be known to compute the joule loss density), the mesh has to be refined adaptively by the method described above. In this way it is possible to obtain meshes that are sufficiently fine to compute the current distributions in solid conductors for all different frequencies.

3. Conclusion

In this paper, a generally applicable technique which allows to maintain a high degree of 'mesh compatibility', required for accurate coupled problem FEM solutions on independendent adaptively refined meshes is presented. The estimated error is transferred to the different subproblem meshes by means of projection techniques. As a consequence, the different errors are combined by using a mathematical averaging operation. The methodology is illustrated on a electromagnetic-thermal coupled problem.

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