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Quasi static field computation by finite elements: Recent developments with respect to the modeling of electrical machines

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1 Introduction

An original design and the step by step optimisation of physical technical devices is in practice often a trial and error process. During the design and construction of a device several expensive prototypes have to be built to monitor and check the mathematical approximations and the physical reality. This procedure is time consuming and expensive. Successful industrial developments demand shorter cycle times to fix or improve the economical competition of particular companies. To effectively compete in the market place nowadays, developed products of higher quality, improved efficiency and better functionality are recommended, leading to devices with very complex geometries. Furthermore, custom designs are becoming very important. The added value of standard massproduction devices is far lower. To solve the techno-economical demands, the idea is to replace the expensive prototyping by numerical simulations.

If an appropriate simulation model is found, various operating points can be simulated on a computer. Even the behaviour of the device for hazardous situations that cannot be measured inside a laboratory and the use of arbitrary even future materials can be studied. The appropriate choice of a calculation technique for an electromagnetic device is always closely linked to the complexity of the problem.

To develop a technical product, parasitic effects such as:

- ferromagnetic saturation
- increased leakage flux
- high operating temperatures
- irreversible flux losses by using permanent magnet materials at elevated temperatures
- coupling between different effects such as thermal-magnetic-mechanical-flow field problems and
- induced currents due to motion effects

have to be considered in the calculations accounting for sufficient accuracy. In devices with complex geometries, those effects can not be treated by a classical analytical approach. Results with a high accuracy are required to predict the behaviour of the technical product. In this case the simulation of the electromagnetic fields and their effects by numerical models is suitable as an appropriate engineering tool. Using computer models and the appropriate numerical algorithms solves the physical problem. The numerical method has to fulfil specified demands such as:

- reliability
- robustness
- application range
- accuracy
- performance.

To see where the numerical simulation finds its place in the analysis of technical devices, Fig.1 shows the links between the real technical device, the classical physical theory and the numerical simulation. This figure makes obvious that the numerical simulation is a connecting element between reality, measurements, and theoretical predictions. As a consequence, all numerical computations represent realistic activities in a fictive laboratory. This means that simulation results should be theoretically measurable in practice. The numerical simulation is in fact an

experiment performed on the computer as a fictitious laboratory, where the engineer is using numerical tools to perform the experiments instead of measurement devices such as current, voltage, power, temperature and force meters.

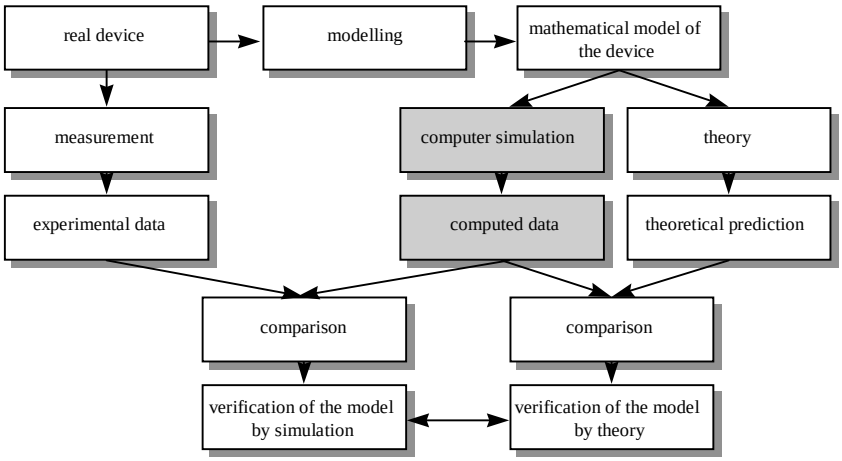


Fig. 1. Theory, experiment and simulation.

The numerical simulation influences the analytical theory where sometimes rough approximations or constants are used to consider physical effects such as ferromagnetic saturation or hysteresis. The verification of numerical solutions and results obtained by the analytical theory can lead to improved analytical models and vice versa. Both numerical simulation and analytical theory help to understand the physical reality and to improve technical predictions.

2 Numerical solution process and analysis

In Fig. 2 the solution process for a system of partial differential equations is outlined.

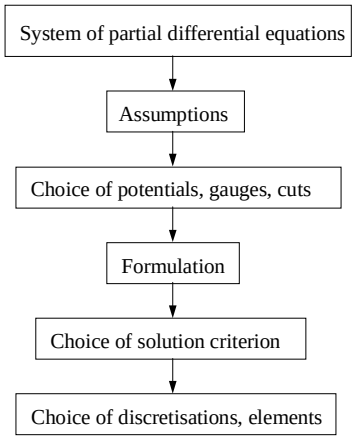


Fig. 2. Solution process for a system of partial differential equations.

The fields are described by differential equations. Assumptions concerning boundary conditions, material properties such as isotropy, dependencies in time, etc. have to be made before a computation of a field can be performed. For example in magneto-static fields, the time derivative is assumed to be zero and therefore no induced currents can be considered.

The choice of the potentials is based on these simplifications. For each problem type, the choice of an appropriate potential is different. The choice of a gauge is necessary to obtain a regular system of equations. Using the finite element method, the choice of the gauge also determines the choice of the element type. However, the

user of a CAD program package that simulates magnetic, electric or thermal fields is usually not involved in choosing for such basic numerical properties.

The numerical method to solve the partial differential equation is understood as a solution criterion. The appropriate solution method depends on the type of equation, such as parabolic, hyperbolic or elliptic. For example, the choice of the elements for the finite element method depends on the differential equation, the potential formulation, and the solution method.

In a two-dimensional magnetostatic problem the unknowns are node potentials. Here, the magnetic vector potential is chosen because the nodal unknowns have only a single component A_Z. In this two-dimensional field problem, the Coulomb gauge is satisfied automatically.

The choice of method for solving a system of linear equations is dependent of the differential problem and its formulation. For example the magneto-static problem is an elliptic differential problem. The Laplace operator is symmetrically adjoint and positive definite. A system of equations with such properties can be solved by a conjugate gradient method.

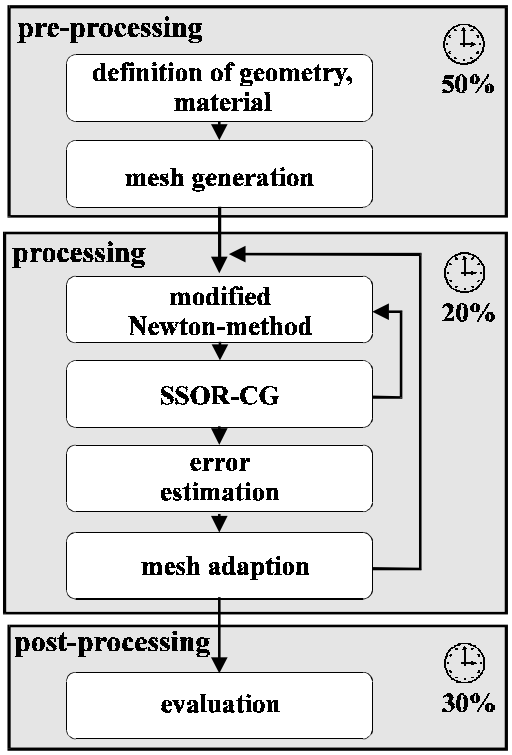


Fig. 3. Solution processes during a field computation session.

To focus on the active parts performed by an design engineer, in principle, field computation is performed in three major steps: pre-processing, processing and post-processing. Fig. 1.3 shows a typical pattern for the FEM approach. The first step consists of the definition of the geometry of the electromagnetic device. Material properties, electrical current densities and boundary conditions are defined. All the activities have to be performed by the design engineer. Therefore, the pre-processing is time consuming. The estimated time expenditure for a two-dimensional problem is given in Fig. 3. The processing, i.e. the solution of a very large system of equations is automatically done in the second step. Only parameters to control the solution process have to be defined by the design engineer. In the last part of the FEM procedure, the interesting field quantities are computed from the solution out of the processing. If the geometrical data can be parameterised, the pre- and post-processing can be automated as well. This represents an important prerequisite for the possibility of the combination of field computation and numerical optimisation.

For designing and constructing electromagnetic devices an accurate knowledge of the field quantities inside the magnetic circuit is necessary. In many cases the air gap is of particular importance (e.g. motors, switches, relays, contactors, actuators). Here the conversion from electrical to mechanical energy and vice versa takes place. In the air gap the field quantities such as flux density and field strength have to be calculated very accurately in order to be able correctly to asses the operational behaviour of the device.

Although Maxwell equations have been known for more than a century, in the past the task in calculating a magnetic circuit was to find as many assumptions and simplifications as possible. Then, results could be obtained with rather low numerical efforts. Using this approach, only devices or problems with a strongly simplified geometry could be studied. It was a design following simple rules, found empirically. Physical effects were considered by correction factors applied to the existing rules. In the following period of time this design through rules has changed into another design philosophy: design analysis. Here, computer models were used to solve the field problem. Analysis means the treatment of the field problem by numerical simulation.

With the ongoing developments in computer hard- and software and numerical research, difficulties concerning computational costs and numerical problems are continuously moving to the background. Today, efficient numerical solutions can be obtained for a wide range of problems beyond the scope of analytical methods. In particular the limitations imposed by the analytical methods, their restrictions to homogeneous, linear and steady state problems can be overcome using numerical methods.

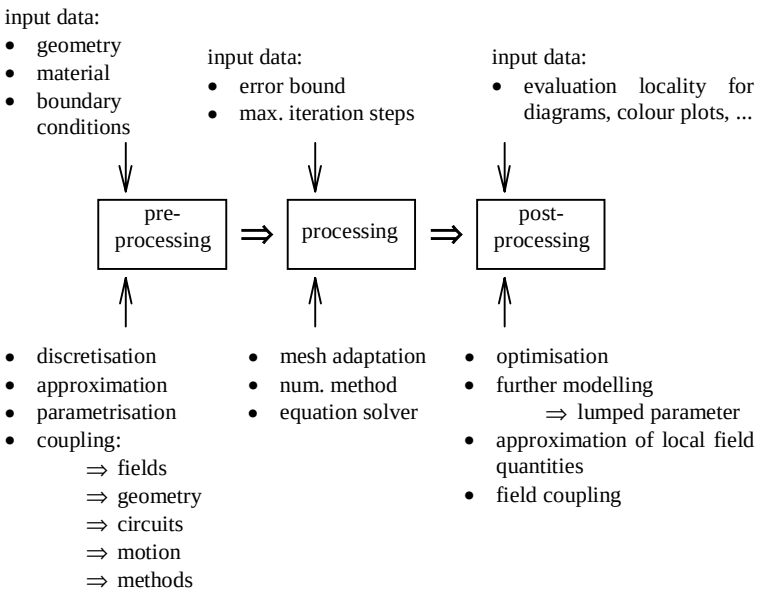


Fig. 4. Field analysis steps.

In general, the procedure for analysing an electromagnetic device can be divided into three steps:

- pre-processing
- processing and
- the post-processing.

In the first step, the field problem is defined and prepared to be solved. The second step delivers the numerical solution of the physical problem. During the post-processing, the obtained solution is prepared to calculate the required field quantities or to evaluate forces and other macroscopic quantities. This threefold approach of defining, solving and evaluating is typical for every analysis procedure, numerical or analytical. The different techniques, data structures or algorithms used in the individual steps, influence and/or limit the overall procedure during the analysis of a field problem (Fig. 4).

To define a field problem, the input data describing the geometry of the domain of interest, the material representation and the boundary conditions are always required. Even with enhanced CAD drawing techniques, most of the analysis time will have to be spent on the pre-processing. Given error bounds will support a desired accuracy of the solution. Often, the user can not influence this step. During the post-processing, the solution must be prepared to study the local field effects. The post-processing represents an open-ended process, because the user of the analysis can evaluate the calculated solution in various ways for different aspects.

The methods and algorithms used in the single steps of the overall procedure can form an efficient analysis or design tool and determine the quality of the results of the analysis. For example a use of particular internal data structure can enable very quick search routines to obtain an efficient, fast and automated discretisation with parameterised geometries and materials. The various possible coupling mechanisms of different fields, circuit equations, methods such as FEM/BEM combinations, motion term or geometries yield into an accurate approximation of a realistic physical problem. The properties of the coefficient matrix decide which equation solver or algorithms must be used to solve the problem.

3 Design strategies

The development and design of electromagnetic devices reflects a complex process. Originating from an initial idea, the construction runs through different phases. This procedure is terminated when a final concept is selected and considered to be designed, subject to various targets and constraints. As a whole, the task of the design engineer is to find solutions for technical problems. On the way to the latter physical and technical product, certain

aspects have to be considered. Technological and material-dependent questions as well as cost effectiveness and ecological constraints have to be taken into consideration. A cut-set of the mentioned boundary conditions controls the feasibility of the final design. With emphasis on electromagnetic devices, Fig. 5 shows a simplified scheme of interdependencies of targets and constraints. This simple pattern clarifies that the design process is strongly dependent on the experience of the engineer and reflects an optimisation procedure with often contradictory aims. Therefore, the necessity of a systematic and strategic design with engineering tools is obvious. Here, solution strategies using modern numerical methods to accelerate and ensure a high-standard technical product in an overall design process are discussed.

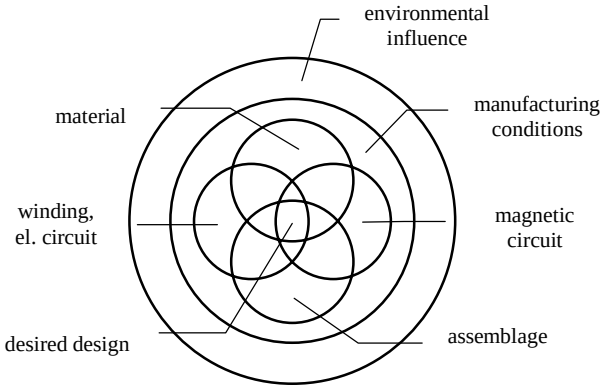


Fig. 5. Interdependencies in the design of electromagnetic devices.

Designing electromagnetic devices includes the calculation and analysis of the electromagnetic field distribution. From the local field quantities forces, torques and losses can be derived to make predictions concerning global quantities such as converted power and efficiency. For complicated geometries analytical field solutions are non-existent or very hard to obtain. Using numerical field computation techniques of a general application range, the microscopic field solution leads via a lumped parameter approach to the desired time-dependent behaviour of the device (Fig. 6).

The microscopic field solution itself delivers important knowledge regarding the material utilisation. Such results offer the opportunity to reduce material, weight and the costs of the latter product. To accelerate development, extensive field computations with various types of material can be performed avoiding expensive prototyping. It is even possible to predict system behaviour before new materials are actually available on the market. With this knowledge, the design engineer can order special material to be developed at the material manufacturer or, vice versa, if the material supplier uses such numerical tools he can suggest and offer the right choice of material for a particular device.

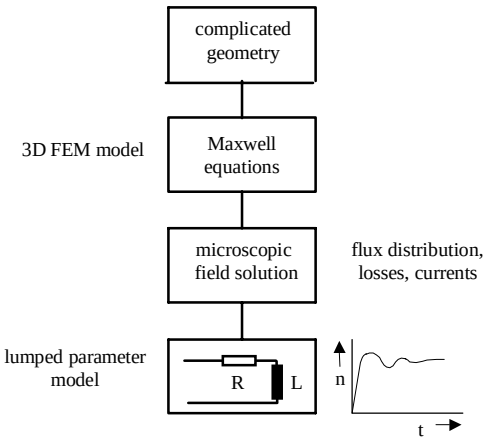


Fig. 6. Analysis scheme using the finite element method.

Lumped parameter models are essential for the development of control strategies for electromechanical devices such as electrical drive systems. To be able to perform real time control schemes, lumped parameter models are used to form an observer control. Here, very accurate field computations are recommended to determine the concentrated elements of such models.

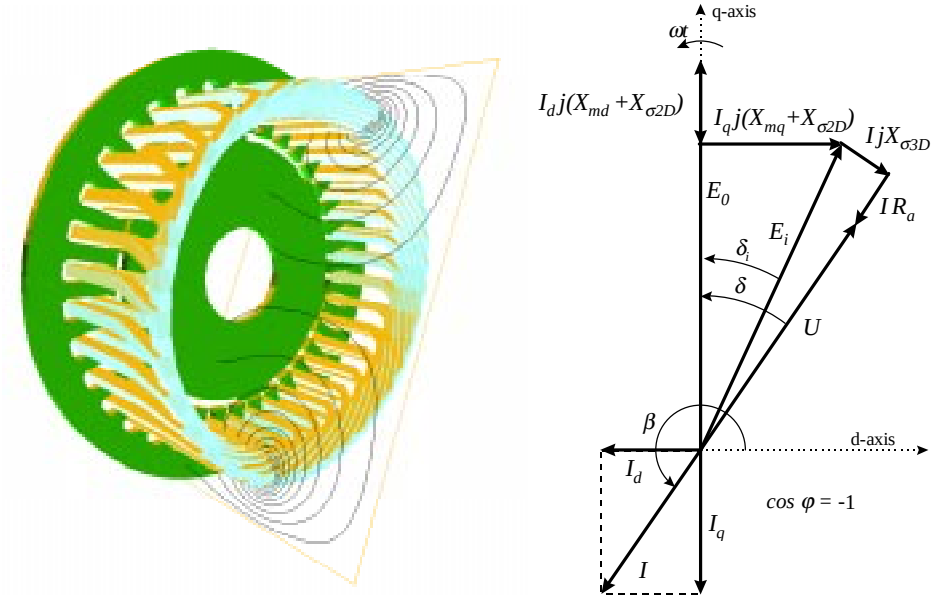


Fig. 7. FEM model of the end-winding area to compute the leakage reactance $X_{\sigma 3D}$ of a servomotor with d-q model in vector diagram representation.

For example, the computation of the leakage reactance of electrical motors can be performed using a three-dimensional FEM model (Fig. 7). The knowledge of this reactance is essential for the optimum control of a permanent magnet-excited servomotor. The vector diagram in Fig. 7 demonstrates the large influence of the leakage reactance $X_{\sigma 3D}$ on the optimum control angle of this permanent magnet machine.

4 Special modeling

Particular problems, such as coupled phenomena, the wish to numerically optimise a device automatically and numerical inaccuracies, require a special treatment of the problem. In the following sections an overview is given to tackle coupled field effects, some aspects with respect to the optimisation of electromagnetic devices are discussed and a method to enhance the accuracy of computed field quantities/forces is introduced.

4.1 Coupled fields

The term “coupled problem” is used in many numerical approaches and applications. Various coupling mechanisms in a different context, such as field problems with electrical circuits, methods in a geometrical or physical sense, couplings in time and/or coupled methods to solve a field problem, are meant with this term. For a proper classification of these problems and related solution methods a systematic definition is proposed. It can be used in the evaluation and comparison of solution methods for various problems.

A coupled system or formulation is defined on multiple domains, possibly coinciding, involving dependent variables that cannot be eliminated on the equation level (Zienkiewicz [1]). In the literature, this notion is often linked to a distinguishing context of various physical phenomena or methods, without further specification. This paper proposes a classification scheme in which the numerical models meeting the proposed definitions can be put. This may lead to the definition of a series of test problems for specified coupled problems and solution algorithms. A classification scheme can simplify the comparison of the various examples and approaches out of the literature that solve such coupled problems.

Next to "coupled problems" the terms "weak-" or "strong-coupled" will be discussed to propose a more homogenous terminology.

To start with a definition of standards or a classification of technical physical problems, the properties and the interdependencies of such phenomena must be considered and discussed.

A general and simplified structure of considered field problems is drawn in Fig. Here, the link between the single fields is determined by material properties depending on the corresponding field quantities. If the field blocks represent numerical methods to solve the single problem in two dimensions, further couplings to external equations such as electrical circuits, magnetic or thermal equivalent circuit models are possible to complete the scheme.

The link between the drawn blocks is, in the context of coupled problems and its numerical solution, a computer model or method. The following question is in which way the physical phenomena have to be considered in an overall solution. From the idea of how to link the effects numerically, a classification of the methods in this sense can be performed.

The coupling of magnetic field equations, described by a partial differential equation (PDE) and the electrical circuit equations providing algebraic expressions for the electrical current densities, can be considered as a special type of coupled problem.

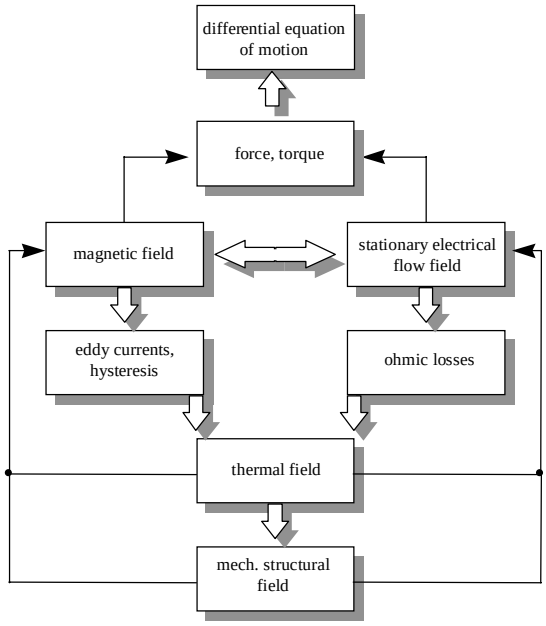


Fig. 8. Simplified structure of coupled fields.

In general, more than one independent physical field variable is involved. The field variables for stationary problems are present in a set of PDEs, or in the transient case in ordinary differential equations (ODE). The coupling is often non-linear and this results in a complicated numerical solution process.

Fields can be described by differential equations. A general form of a differential equation has to be studied to understand the parameter coupling between equations. Equation (1) represents the general form of a differential equation with its possible coefficients in the particular terms. In coupled fields problems, such coefficients are field dependent and represent the link between the various field types, such as magnetic/thermal etc.

$$a\frac{\partial}{\partial t}f \quad -\nabla\cdot(\lambda\nabla f) \quad +\gamma v\nabla f \quad +\alpha f \quad = g \tag{1}$$

5. source
4. absorption
3. convection
2. diffusion
1. parabolic, transient term

The first term characterises the equation being parabolic. Stationary equations do not contain this term ($\partial/\partial t = 0$). In Laplace's equation terms 2 and 5 are present. To obtain the Helmholtz equation, term 4 can be added. For these two types of equations, a variational formulation exists. The 3rd term is typical for problems considering motion effects.

The coefficients in (1) are usually derived from given material characteristics. For example, temperature dependent material properties of permanent magnet material can be used to define a coupled magnetic/thermal field problem. Within a field problem definition, the characteristics vary locally.

In general, it is possible to distinguish between the coupled problem in two ways, in its physical or its numerical nature. Very often a coupled problem is called either

- strong, or
- weak.

In the physical sense, the strong coupling describes effects that are physically strongly coupled and the phenomena can not numerically be treated separately. If numerical formulations exist, the coupling can be found in the governing differential equations due to the coupling terms. The weak coupling describes a problem where the effects can be separated. The problem with this definition is obvious: If coupled problems are studied, it is not very well known how strong or weak they are physically coupled; this is the desired answer expected from the analysis of the overall problem. For example if the material property describing parameters are non-linearly dependent on the field quantities, the coupling, (strong/weak) can even change with varying field quantities and the field quantities are the result of the analysis. Therefore, the definition of strong/weak coupling should be chosen according to the numerical aspects instead of their physical nature. Choosing for the numerical aspects, it is possible to have a combined strong/weak coupling of field problems. This means that the strategy of coupling can vary, and thus the methods/models, while solving the problem.

Numerical strong coupling is the full coupling of the problem describing equations on matrix level. The equations of all involved and modelled effects are solved simultaneously. This implies that the coupling terms are entries in the coefficient matrix as well.

The numerical weak coupled problem is understood as a cascade algorithm, where the considered field problems are solved in successive steps and the coupling is performed by up-dating and transferring the field dependent parameters to the other field definition before solving again.

Since the problems cannot be distinguished by means of elimination, a bi-directional influence exists. The sensitivity of a sub-problem to changes of the variables of the studied problem can differ strongly. It is difficult to quantify a threshold for separation of both groups, and therefore the separation may be considered as somewhat subjective. In this respect, the time constants of the sub-problems play an important role. Usually the thermal and mechanical time constants are several orders larger than the electromagnetic time constants. So, on a short term, the problem with a larger time constant can be considered as weak coupled. But this is not true if the stationary solution is of interest.

In the following, the strong coupling FEM equation system of a magnetic/thermal problem is derived. For simplicity it is assumed that both field problems are defined on the same mesh. For a more realistic coupling, projection methods can be applied to enable the field definitions on different meshes. This approach results in additional coupling terms in the final coefficient matrix. For further simplicity, the material's properties ν and k are assumed to be independent of A and T respectively. The coupling of the fields causes the remaining non-linearity by the loss mechanism.

The magnetic/thermal coupled problem is modelled by a set of two equations:

$$\begin{aligned} \upsilon\nabla^2 A - j\omega\cdot\sigma A &= -J_0 \\ k\nabla^2 T &= -q_0 \quad . \end{aligned} \tag{2}$$

It is assumed that the source term of the thermal equation consists only of joule losses:

$$q_0 = q_{joule} = \rho\cdot J^2(A) = \rho\cdot\left(J_0^2 + \frac{\omega^2 A^2}{\rho^2}\right) \quad . \tag{3}$$

The first term will appear on the right-hand side of the system. The second term, the eddy current losses, have to be linearised and represent the coupling term with a non-linear coefficient:

$$q_0 = \rho J_0^2 + \left(\frac{\partial q_{eddy}}{\partial A}\right) A = \rho J_0^2 + \left(2\sigma\omega^2 A\right) A = \rho J_0^2 + m(A)A \tag{4}$$

Written in matrix/vector notation eq.(4) is rewritten as:

$$\begin{bmatrix} \upsilon\nabla^2 - j\omega\cdot\sigma & 0 \\ m & k\nabla^2 \end{bmatrix} \cdot \begin{bmatrix} A \\ T \end{bmatrix} + \begin{bmatrix} J_0 \\ \rho J_0^2 \end{bmatrix} = 0 \quad . \tag{5}$$

There is a coupling present through the coefficients, although there is a zero entry in the off-diagonal of the magnetic equation. Applying the Galerkin approach results in an integral per element of the form:

$$\int_{\Omega_e} \left(\begin{bmatrix} N_A \\ N_T \end{bmatrix}^T \cdot \begin{bmatrix} \upsilon\nabla^2 - j\omega\cdot\sigma & 0 \\ m & k\nabla^2 \end{bmatrix} \cdot \begin{bmatrix} N_A \\ N_T \end{bmatrix} \cdot \begin{bmatrix} A_i \\ T_i \end{bmatrix} + \begin{bmatrix} N_A \\ N_T \end{bmatrix}^T \cdot \begin{bmatrix} J_0 \\ \rho J_0^2 \end{bmatrix} \right) d\Omega = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad . \tag{6}$$

For two-dimensional first order elements this yields six algebraic equations:

$$\begin{bmatrix} K_A & 0 \\ M & K_T \end{bmatrix} \cdot \begin{bmatrix} A_i \\ T_i \end{bmatrix} - \begin{bmatrix} F_A \\ F_T \end{bmatrix} = \begin{bmatrix} * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & 0 \\ + & + & + & * & * & * \\ + & + & + & * & * & * \\ + & + & + & * & * & * \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ T_1 \\ T_2 \\ T_3 \end{bmatrix} + \begin{bmatrix} \oplus \\ \oplus \\ \oplus \\ \otimes \\ \otimes \\ \otimes \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad . \tag{7}$$

The first three equations are complex, the last three real. The entries marked with an * are the same terms that would be found in the de-coupled problem. The terms marked with a + result from the eddy current heat source term.

4.2 Numerical optimization

The design process of electromagnetic devices reflects an optimisation procedure. The construction and step by step optimisation of technical systems in practice is a trial and error-process. This design procedure may lead to sub-optimal solutions because its success and effort strongly depends on the experience of the design engineer (Fig. 9).

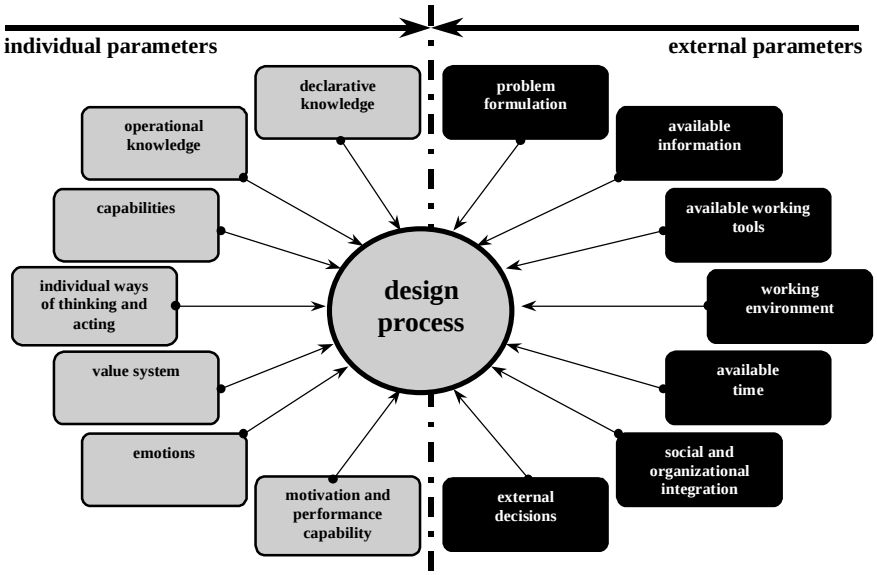


Fig. 9. Parameters affecting the design.

To avoid such individual parameters and thus to achieve faster design cycles, it is desirable to simulate the physical behaviour of the system by numerical methods. In order to get an automated optimal design, numerical optimisation is recommended to achieve a well defined optimum.

Optimisation of electromagnetic devices turns out to be a task of increasing significance in the field of electrical engineering. The term of Automated Optimal Design (AOD) describes a self-controlled numerical process in the design of technical products (Hameyer [2]). Recent developments in numerical algorithms and more powerful computers offer the opportunity to attack realistic problems of technical importance (Pahner [3]).
The distinctive feature of this type of optimisation problem is its complexity, which results from a high number of design parameters, a complicated dependence of the quality on design parameters and various constraints. Often the direct relation of the desired quality of the technical product on the objective variables is unknown. Stochastic optimisation methods in combination with general numerical field computation techniques such as the finite element method (FEM) offer the most universal approach in AOD. This section discusses methodology, characteristic features and behaviour of optimisation methods.

To be able to select the appropriate optimisation algorithms to form an overall design tool together with the numerical field computation, the properties of typical electromagnetic optimisation problems will be discussed (Rao [4], Pahner [3]). Electromagnetic design and optimisation problems reflect mainly the following categories:

- constrained
- problem type:
  ☒ parameter- or static optimisation, $f(\mathbf{x}) \rightarrow \min$.
  ☐ trajectory, or dynamic problem, $f(\dot{\mathbf{x}}, \mathbf{x}) \rightarrow \min$.
- non-linear objective function
- design variables:
  ☒ real
  ☐ mixed real/integer
- multi-objective function
- interdependencies of the quality function and the design variables are unknown; no derivative information available
- the quality function is disturbed by stochastic errors caused by the truncation errors of the numerical field computation method.

In reality electromagnetic optimisation problems are constrained due to the various reasons (Fig. 5). Nowadays optimisations are performed mainly as static problems. Numerical optimisations require huge amounts of computation time. Therefore, the optimisation as aimed at here, combined with the FEM, of the dynamic system behaviour is not yet performed. For transient problems an evaluation of the quality function by numerical methods (FEM) is too time consuming. Considering mixed real/integer design variables results in long computation times as well. The tick boxes in the list that are not marked, represent developments for the future. The optimisation problems that can be solved will grow with increasing computer performance as well.

In general, optimisation means to find the best solution for a problem under the consideration of given constraints and it does not mean to select the best out of a number of given solutions. In other words the definition of an optimum is:

Define a point $\mathbf{x}_0=(x_1, x_2, ..., x_n)^T$ with the independent variables $x_1, x_2, ..., x_n$ in such a way that by their variation inside the admissible space the value of a quality function $Z(\mathbf{x}_0)$ reaches a maximum or a minimum. The point $\mathbf{x}_0$ is described as the optimum.

This definition in mathematical terms:

Minimise a quality function

$$Z(\mathbf{x}) = Z(x_1, \ldots x_n) \rightarrow \min .$$

considering

$$\begin{aligned} g_j(\mathbf{x}) &\le 0 \quad j = 1(1)m \\ h_j(\mathbf{x}) &= 0 \quad j = 1(1)p \quad . \end{aligned} \tag{8}$$

The g_j are called inequality and the h_j equality constraints. Any constraint can be determined in one of these forms. Constraints represent limitations on the behaviour or performance of the design and are called behaviour or

functional constraints, whereas physical limitations on the design variables (e.g. availability, manufacturability) are known as geometric or side constraints. If an optimisation problem with only inequality constraints $g_j(\mathbf{x}) \le 0$ (Fig. 10) is considered, all sets of values $\mathbf{x}$ that satisfy the equation $g_j(\mathbf{x}) = 0$ form a (N-1)-dimensional hyper-surface of the design surface, the constraint surface. The constraint surface splits the design surface into two basic regions: the feasible or acceptable region with $g_j(\mathbf{x}) \le 0$, and the infeasible or unacceptable region with $g_j(\mathbf{x}) > 0$. If, during the progress of the optimisation, a design vector lies on a particular constraints surface, this constraint is called an active constraint.

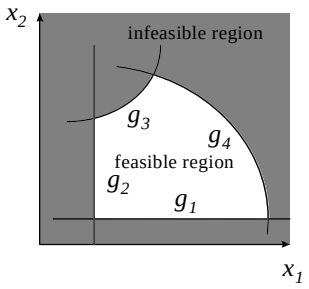


Fig. 10. Constraint surfaces in a hypothetical two-dimensional design space, with side constraints (g_1 and g_2) and behaviour constraints (g_3 and g_4).

The independent variables are the design parameter or object variables. Fig. 11 shows the shape of a two-dimensional quality function with the global optimum and difficulties such as saddle points and local extremum.

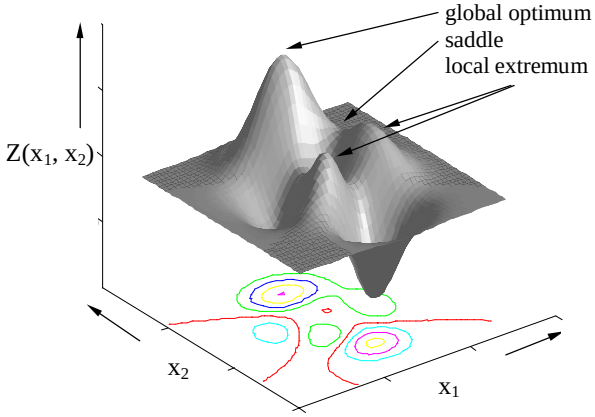


Fig. 11. Quality function with two object variables.

To obtain commensurable criteria for the generation of the design variations and to support a simplified formulation of the stopping criteria of the algorithm, the design variables should be transferred into a normalised form:

$$x_j = \frac{x_{j,d} - x_{j,l}}{x_{j,r}} \quad \text{or} \quad x_j = \frac{x_{j,d} - x_{j,0}}{x_{j,0}} \tag{9}$$

where $x_{j,d}$ is the original parameter with its given physical dimension, $x_{j,l}$ the lower bound of the parameter variation range, while $x_{j,r}$ denotes the actual parameter variation range. If no lower or upper bound of the parameter is given, the design variable can be normalised to its initial value $x_{j,0}$.

The appropriate formulation of the quality function represents a particular problem. All design aims must be formulated in this single function and all object variables must be implemented. Multiobjective optimisation extends the optimisation theory by permitting multiple objectives to be optimised simultaneously. It is also known under different names, such as Pareto optimisation, vector optimisation, efficient optimisation, multicriteria optimisation, etc. One way of formulating a single objective function is a weighted linear combination of the q different objective functions:

$$f(\mathbf{x}) = \sum_{i=1}^{q} \gamma_i f_i(\mathbf{x}) \tag{10}$$

where γ_i denotes a weighting factor best formulated with the properties

$$\gamma_i \in IR, \quad 0 < \gamma_i < 1, \quad \sum_{i=1}^{q} \gamma_i = 1 \tag{11}$$

and $f_i(\mathbf{x})$ are the individual objective functions. In practice, the choice of the weighting factors may already influence the result of the optimisation. It is often not straightforward to select a single fixed weighting factor for each objective, especially if the objective function is erroneous or if no particular preference is given to one of the objectives.

4.2.1 Methods

In general, numerical optimisation algorithms are iterative methods, constructed to reach the desired optimum in successive steps. This is performed following particular rules to vary the object variables and to determine the search direction. The various algorithms differ only in the choice of step-length, determination of the search direction and in the choice of a stopping criterion. A general form of an optimisation algorithm can be given by applying:

step 0: Choose a start-vector $\boldsymbol{x}^{(0)}$ in the admissible space and set the counter of iteration k=1.
step 1: Evaluate the solution-vector according to a quality function.
step 2: Check whether a stopping criterion is fulfilled. If yes, stop the optimisation; if not, set k=k+1.
step 3: Generate a new solution-vector by variation of the objective variables using a suitable step-length and search direction. Continue with step 1.

With the given properties of the electromagnetic optimisation problems, the requirements of the optimisation algorithms can be formulated. Numerical methods have to be examined with regard to the following criteria:

- reliability
- robustness
- insensibility to stochastic disturbances
- application range
- accuracy
- stable solutions
- performance.

Optimisation algorithms can be classified into:

- deterministic or stochastic and
- direct or indirect methods.

Deterministic methods are basically local optimisation methods, often based on the construction of derivatives or approximations of the derivative of the objective function (Fletcher [5], Bertsekas [6]). Such gradient based methods, e.g. Conjugate Gradient (CG), Newton, Quasi Newton, Broyden-Fletcher-Goldfarb-Shanno (BFGS), etc. are very popular, as they are effective and converge to the local optimum in a small number of steps. This low number of quality function evaluations would be ideal when applying a computationally rather expensive FEM analysis to evaluate the objective function. If no analytical objective function exists or the derivative is difficult to obtain, the use of these methods is not appropriate. Furthermore, these methods are very sensitive to stochastic disturbances, especially present in the derivative information they are based upon. Most deterministic methods additionally require the transformation of a constrained optimisation problem into an unconstrained one. In the case of a multimodal objective function, as is often the case in multiobjective optimisations, these methods are unable to find the global minimum (optimum).

An effective approach to compute the sensitivity information during a FE-analysis is introduced to field computation by Park et al. [7, 8]: the method of adjoint variables. This method was previously successfully applied in electronic circuit optimisation. Here, the sensitivity of the objective function with respect to a set of design parameters can be computed with only two solutions.

In general, the human interaction involved in formulating an optimisation problem, in particular in finding the derivatives, is a considerable economical factor when evaluating the efficiency of any optimisation method. The preparation for such an optimisation task might require weeks, while the execution of the actual optimisation run is a matter of minutes. Over the past years, research has been carried out for achieving automatic differentiation of computer codes. The idea is to provide first and higher order derivatives of coded vector functions, without human

interaction. A variety of automatic differentiation software is already available, such as *ADIC* (Bischof et al. [9]), *ADOL-C* , *PCOMP* , etc. At present, software code contained in a single file with up to 10.000 lines of code can be automatically differentiated.

Stochastic optimisation methods, on the contrary, such as simulated annealing, evolution strategy and genetic algorithms, do not require derivative information. Any kind of design constraint can be implemented in a simple manner, by just rejecting a design that violates any constraint or by using penalty terms in combination with the objective function. These methods are capable of handling large dimensional optimisation problems and are less sensitive to stochastic disturbances of the objective function value. The major drawback of these methods is the large number of function evaluations required when compared to deterministic methods. This fact has, in a first view, an even greater impact when considering FEM based objective function evaluations. The first combinations of the finite element technique and stochastic optimisation methods considered partial models. One of the first publications reporting the application of a stochastic method to optimise an entire electrical machine is reported by Hameyer [2]. Since then, a large variety of optimisation problems have been solved using the combination of stochastic methods and finite element function evaluation (Palko [10]). Although the plain execution time of such optimisations is large when compared to deterministic approaches, the simplified set-up of the optimisation task and their ability to find the global optimum make such an overall optimisation procedure attractive.

The rather high computational expense of the FEM has always resulted in attempts to reduce the number of function evaluations by applying statistical methods to sample the search space efficiently. A variety of methods can be entitled as indirect, as the optimisation algorithms are executed on an approximation of the real objective function. The combination of the Response Surface Methodology (RSM) and Design of Experiments offers a whole set of statistical tools not only to optimise a design, but also to evaluate the main and interactive effects of the design parameters. Only a few applications of this method have been reported in electromagnetics in conjunction with FEM function evaluations. A major drawback of these methods is the fact that due to the use of first or second order (global) polynomials, there is only a remote possibility of finding the global optimum in a search space with several local optima. This problem has recently been relaxed by the application of radial basis functions for the approximation of objective functions. The first applications, employing the so-called General Response Surface Method (GRSM) have been introduced to the electromagnetics community by Alotto et al. [11]. The experience has shown, however, that these methods are applicable to rather low dimensional problems only, as their practical efficiency deteriorates with a high number of design variables. Other methods utilise the derivative information made available by the approximation based on radial basis functions. These methods increase the probability of finding the global optimum present in the approximation.

4.3 Force computation

Using arbitrary potentials instead of physical quantities and the associated functionals in the formulation of the equations, raises the need for a closer look at the post-processing. The user of a FEM system desires to analyse a physical system in terms of field strength, energies, forces, densities etc. The potential itself does not necessarily have a physical meaning. In some cases, such as in the electrostatic and in the thermal analysis, the potential represents the electric potential and the temperature respectively (Table 1). Therefore, most of the interesting quantities in the post-process are numerically derived quantities. The type and order of the shape function of the potential over an element (linear, quadratic, etc.) and the element type (nodal, edge, etc.) determine the achievable relative accuracy of numerically derived values. The accuracy of the results is influenced by the discretisation and, related to it, the choice of the error estimator for an adaptive mesh refinement, if applied. Another difficulty arises in the calculation of lumped parameters (inductances, reactances, etc.), used in non-FEM analysis procedures, such as circuit analysis. Several different definitions of these quantities may exist, as for the inductance calculation of linear and non-linear energy transducers.

The aim of this chapter is to provide an overview of possible derived quantities, the necessary formulations and ways of influencing the accuracy of the results. The chosen potentials for the different types of problems do not necessarily directly represent a physical quantity. The formulations for defining these potentials are chosen such that their application might impose simplifications in the formulation of the functionals or the choice of the gauges. A selection of problem types and the physical meaning of their potentials are collected in Table 1.

Table 1. Physical meaning of selected potentials.

Type of analysis	differential equation	type of potential	physical meaning
electrostatic	$\varepsilon\nabla^2 V = -\rho$	scalar	electric potential
magnetostatic	$\nu\nabla^2 A = -J_0$	vector	none, $\mathbf{B} = \nabla\times\mathbf{A}$
thermostatic	$\lambda\nabla^2 T = Q$	scalar	temperature
time-harmonic magnetic	$\nu\nabla^2 A - j\omega\sigma A = -J_0$	vector	induced currents related to A

The achievable accuracy of all derived values cannot be better than the accuracy of the computed potentials. The latter is determined by the choice of the element type, the shape function and the functionals used.

Most local field quantities, as well as other derived quantities such as force, require numerical derivatives of the potentials. Using nodal elements, the potentials are known at each node as a result of the approximate solution of the partial differential equation. The change of the potential inside one element is determined by the choice of the shape function:

$$A = a + bx + cy \quad . \tag{12}$$

Knowing the potentials at the nodes of the elements, the coefficients a, b and c can be calculated using this basis function. The definition of the potential now determines the required mathematical operations yielding the required local field value. In two-dimensional magnetostatic problems, the vector potential $\mathbf{A}$ is defined by:

$$\mathbf{B} = \nabla\times\mathbf{A} \quad . \tag{13}$$

Using such linear shape functions to approximate the vector potential, the x- and y-components of the flux density inside a finite element are calculated as follows:

$$\begin{aligned} B_x &= \frac{\partial A}{\partial \mathrm{y}} = \frac{1}{2\Delta_e}\sum_i^3 c_i A_i = \text{const.} \\ B_y &= -\frac{\partial A}{\partial x} = -\frac{1}{2\Delta_e}\sum_i^3 b_i A_i = \text{const.} \end{aligned} \tag{14}$$

The flux density B inside an FEM model is piecewise constant (**Erro! A origem da referência não foi encontrada.**) if a continuous distribution of the vector potential is assumed. Accounting for this and assuming a small value of h as the maximum characteristic diameter of a finite element, the FEM is convergent towards the exact solution of order q+1. The constant q describes the polynomial order of the elements used. With ε as the global error, the order of convergence for the potential solution is

$$\|\varepsilon\| \le C \cdot h^{q+1}. \tag{15}$$

The factor C is independent of the size h of the elements and depends only on the

- type of discretisation
- choice of shape function
- smoothness of the exact solution.

Equation (15) identifies the convergence problem transferred into the approximation problem (Hameyer [12]). Using first order linear shape functions the rate of convergence is of order $O(h^2)$. Deriving the field quantities from the potential formulation numerically results in a rate of convergence $O(h)$ for those quantities, i.e. a loss in accuracy of one order compared to the potential solution. Using these field quantities this inherent inaccuracy influences the results of force calculations. This fact identifies the difficulty in obtaining accurate field quantities as a problem of the order of convergence of the numerical method used. To illustrate this fact, consider a domain containing a single linear material. By applying Dirichlet boundary conditions of different values to the left and the right domain border, a constant flux is imposed (Fig. 12).

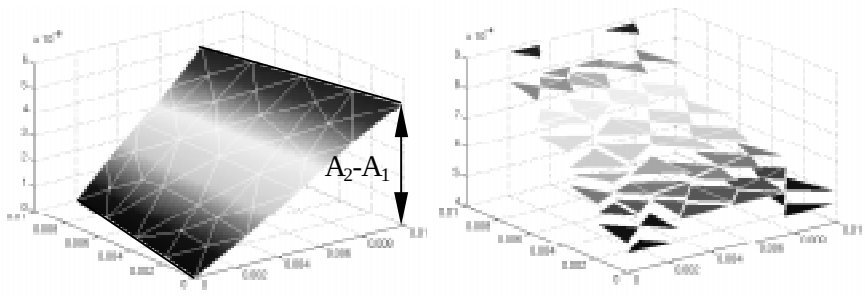


Fig. 12. a) Continuous vector potential; b) piece wise constant flux density.

The loss of one order of accuracy due to the numerical differentiation is inherent and effects all quantities based on such values. A possible way to increase the accuracy of local field quantities is discussed here: re-calculation of the field distribution in parts of the domain by a local post-process.

Here, it will be focused on the practical application of the static electromagnetic field solution of Laplace's equation in a local post-process to increase the accuracy of an existing solution obtained by the standard finite element method using first order elements. Advantages and drawbacks are discussed.

The Laplace equation $\nabla^2 A = 0$ in two dimensions, expressed in polar co-ordinates (r,Φ) is:

$$\frac{\partial}{\partial r}\left(r\frac{\partial A}{\partial r}\right)+\frac{1}{r}\frac{\partial^2 A}{\partial \Phi^2}=0 \quad \Rightarrow \quad r^2\frac{\partial^2 A}{\partial r^2}+r\frac{\partial A}{\partial r}+\frac{\partial^2 A}{\partial \Phi^2}=0 \qquad (16)$$

Assuming linearity and uniformity, and applying a Fourier series to eq.(16), yields the harmonic function:

$$A(r,\Phi)=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty} r^n\left\{\alpha_n\cos(n\Phi)+\beta_n\sin(n\Phi)\right\} \qquad (17)$$

with its coefficients:

$$\alpha_n=\frac{1}{\pi R^n}\int_0^{2\pi} A(R,\Phi)\cdot\cos(n\Phi)\cdot d\Phi \quad , \quad \beta_n=\frac{1}{\pi R^n}\int_0^{2\pi} A(R,\Phi)\cdot\sin(n\Phi)\cdot d\Phi \qquad (18)$$

The procedure for solving eq.(17) describes the solution of a Dirichlet problem on a circle with given boundary values at its circumference. The coefficients α_n and β_n can be calculated using known potentials $A = A(R,\Phi)$ at the circumference of a circle with radius *R*.

Now a finite number of *N* equi-angularly arranged points are applied onto the circumference of the circle.

$$A_i(R,\Phi_i)=A(R,i\cdot\tfrac{2\pi}{N}) \qquad i=1(1)N \qquad . \qquad (19)$$

With *N* boundary potential values u_i known on the circumference and according to the properties of harmonic functions the first term in eq. (17) can be written by:

$$A\big|_{r=0}=\frac{\alpha_0}{2}=\frac{1}{N}\sum_{i=1}^{N} A_i \qquad . \qquad (20)$$

The Fourier coefficients are rewritten as follows:

$$\alpha_n=\frac{2}{N\cdot R^n}\sum_{i=1}^{N} A_i\cos(n\Phi_i) \quad , \quad \beta_n=\frac{2}{N\cdot R^n}\sum_{i=1}^{N} A_i\sin(n\Phi_i) \quad . \qquad (21)$$

With the Fourier series (17) and their coefficients eq.(21) the potential in the centre of a circle can be computed knowing only the boundary potential values on the circumference of the circle.

Using this approach inside a finite element solution, the value of the potential of a field point now depends on the solution in several finite elements. Thus, local numeric errors in single elements have a relatively small

influence on the solution in the considered field point. Applying (17) derivatives at the centre of the circle can be calculated in a closed analytical form, avoiding numerical differentiation.

$$\left.\frac{\partial A}{\partial y}\right|_{r=0} = \beta_1 = B_x = \frac{2}{N \cdot R}\sum_{i=1}^{N} A_i \sin\Phi_i$$
$$\left.\frac{\partial A}{\partial x}\right|_{r=0} = \alpha_1 = B_y = \frac{2}{N \cdot R}\sum_{i=1}^{N} A_i \cos\Phi_i \qquad (22)$$

The idea is to adapt the described process, of solving a Dirichlet problem on a circular surface, to determine the vector potential in a point P_1 of a discretised finite element domain (Fig. 13). R_1 is the radius of the considered circular surface and the dots at the circumference indicate the points of known vector potential values computed beforehand. These points do not have to be nodes of the actual finite element mesh. This feature makes the technique very advantageous to automatic and adaptive meshing schemes in which the user can not guarantee the control of the mesh and especially its symmetry.

To obtain the potential distribution at a given contour inside a finite element domain, multiple circles have to be evaluated. Overlapping circles guarantee a continuous solution in the considered region after the post-process.

The numerical shape of (17), (20) and (21) enables an easy implementation of the procedure in a finite element program package. The derivatives in the centre of the circle are represented by the Fourier coefficients. Thus, no additional computational effort is necessary to compute the flux density in the centre.

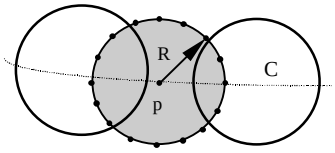
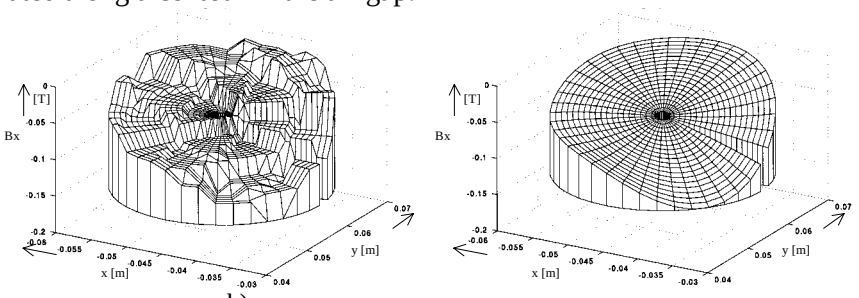


Fig. 13. Multiple circles to determine the vector potential on a contour.

The local solution of the Laplace equation inside an air gap of an electromagnetic device, using a Fourier series approximation for the vector potential, results in a significant increase in accuracy of the derived field quantities. To compare the results obtained by the local field evaluation to the conventionally obtained field quantities of first order elements, Fig. 14 shows the computed magnetic flux density derived by $B = \nabla \times A$ and B_x using the Laplace approach. For this application of the local Dirichlet problem, 24 potential boundary values on the circumference of the circle were used.

Applying such field quantities to the Maxwell stress tensor to compute the local forces acting on bodies inside an electromagnetic field yields values of higher accuracy. To obtain the torque of an electrical machine, the local force values are integrated along a contour in the air gap.



a) b)

Fig. 14. a) Computed vector potential inside the circular FEM domain and b) the resulting flux density B_x derived by applying the local post-process.

Another approach to compute the torque more accurately, uses the values of the magnetic vector potential on two concentric circles with radii R_i and R_o as boundary conditions (Fig. 15). Local field values on the circular contour C with radius $R_i < r < R_o$ are calculated.

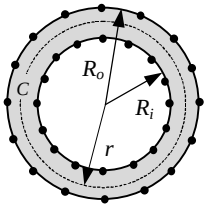


Fig. 15. Local Dirichlet problem for a cylindrical air gap.

If the inner radius R_i is taken as a reference, the general solution of Laplace's equation is:

$$A(r,\Phi)=\sum_{k=1}^{N}\left(a_k\left(\frac{r}{R_i}\right)^k\cos(k\Phi)+b_k\left(\frac{r}{R_i}\right)^k\sin(k\Phi)+c_k\left(\frac{R_i}{r}\right)^k\cos(k\Phi)+d_k\left(\frac{R_i}{r}\right)^k\sin(k\Phi)\right) . \tag{23}$$

The coefficients a_k, b_k, c_k and d_k are independently determined for each circular harmonic. A fast Fourier transformation (FFT) algorithm is used to express the magnetic vector potential at the boundaries as a series of such circular harmonics:

$$A(R_i,\Phi)=\sum_{k=1}^{N}\left(a_{k,i}\cos(k\Phi)+b_{k,i}\sin(k\Phi)\right) ,$$
$$A(R_o,\Phi)=\sum_{k=1}^{N}\left(a_{k,o}\cos(k\Phi)+b_{k,o}\sin(k\Phi)\right) . \tag{24}$$

$$\begin{bmatrix}1 & 1\\ \left(\frac{R_o}{R_i}\right)^k & \left(\frac{R_i}{R_o}\right)^k\end{bmatrix}\begin{bmatrix}a_k\\ c_k\end{bmatrix}=\begin{bmatrix}a_{k,i}\\ a_{k,o}\end{bmatrix} ,$$
$$\begin{bmatrix}1 & 1\\ \left(\frac{R_o}{R_i}\right)^k & \left(\frac{R_i}{R_o}\right)^k\end{bmatrix}\begin{bmatrix}b_k\\ d_k\end{bmatrix}=\begin{bmatrix}b_{k,i}\\ b_{k,o}\end{bmatrix} . \tag{25}$$

Once the magnetic vector potential at the contour C is known, the normal and tangential component of the magnetic flux density can be determined:

$$B_n(r,\Phi)=\sum_{k=1}^{N}\left(-ka_k\frac{r^{k-1}}{R_i^k}\sin(k\Phi)+kb_k\frac{r^{k-1}}{R_i^k}\cos(k\Phi)-kc_k\frac{R_i^k}{r^{k+1}}\sin(k\Phi)+kd_k\frac{R_i^k}{r^{k+1}}\cos(k\Phi)\right) ,$$
$$B_t(r,\Phi)=\sum_{k=1}^{N}\left(-ka_k\frac{r^{k-1}}{R_i^k}\cos(k\Phi)-kb_k\frac{r^{k-1}}{R_i^k}\sin(k\Phi)+kc_k\frac{R_i^k}{r^{k+1}}\cos(k\Phi)+kd_k\frac{R_i^k}{r^{k+1}}\sin(k\Phi)\right) . \tag{26}$$

The tangential force component F_t results in the torque T of the device. It can be shown (Salon 13, Mertens et al. 14) that the value of the torque is given by

$$T=\frac{2\pi}{\mu_0}\sum_{k=1}^{N}\left(k^2\left(b_kc_k-a_kd_k\right)\right) , \tag{27}$$

being independent of the radius r of contour C. It is not necessary to calculate the normal and tangential component of the magnetic flux density on the contour resulting in a faster algorithm, when the overall torque is aimed at. The

proposed method can easily be extended to time-harmonic problems. If all values are rms-values the torque is obtained by adding the torque calculated using the real- and the imaginary-component of the solution.

$$T = T_{real} + T_{imag} \quad . \tag{28}$$

This method has its advantage for electrical machine analysis as it is suited for small air gaps. By using eq.(28) the torque is evaluated directly, without the explicit calculation of the flux densities.

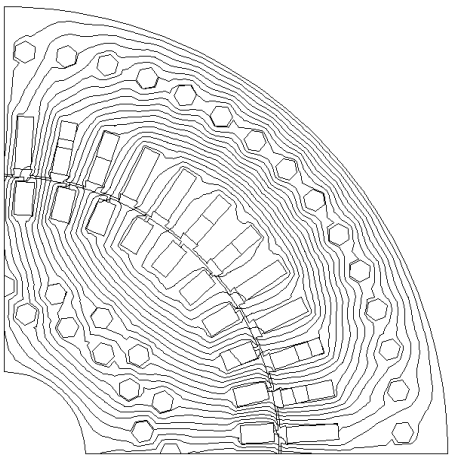


Fig. 16. Equipotential plot of the real component solution of a 400 kW induction motor.

The performance of this method is compared with the classical Maxwell stress tensor method using a model of a 400 kW induction machine for tests.

To ensure accurate results, a good trade-off between mesh refinement and using the enhanced post-processing methods is necessary. A relatively coarse discretisation in the air gap of the induction machine was chosen (Fig. 17). With such a coarse discretisation, the Laplace-based method is less sensitive to the actual choice of the contour inside the air gap than the classical method. The air gap spans a region between an inner radius of 0.186 m up to an outer radius of 0.1875 m. Fig. 18 shows the variation of the calculated torque. Contours with different radii are chosen. For the Laplace-based method, the inner and outer radii are varied simultaneously. Therefore, the value of the torque varies symmetrically towards the middle of the air gap.

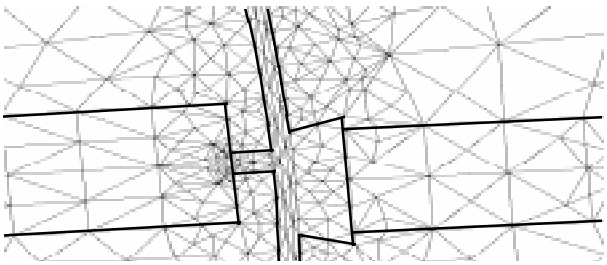


Fig. 17. Detail of the discretisation in the air gap.

The variation of the calculated torques using the enhanced method is much smaller when compared to the classical method. It must be stated, however, that an appropriate mesh refinement scheme would lead to better results even for the classical torque computation.

The same basic idea, as used in the ‘circle’ approach, yields the local solution for the three-dimensional field. The local field problem is now defined by the known potential values equally distributed along the surface of a sphere assumed to be the boundary potential values of the local field problem (Hameyer [12]).

As a result, in Fig. 19 the quadratic convergence, referred to the characteristic length h of a finite element, of the FEM potential solution and the rate of convergence of the force computations using both the classical and the new post-processing approach, is plotted versus the number of tetrahedron elements. The same statement can be made for the two-dimensional approach. The triangles in Fig. 19 indicate the theoretical gradient of convergence eq.(15). The gradient-triangles in Fig. 19 indicate the theoretical rate of convergence for the quadratic and the linear

convergence case. It can be seen as theoretically expected that the relative error in an energy norm of the FEM potential solution converges quadratically, referred to the specific diameter *h* of the elements eq.(15), by increasing the number of first order tetrahedron elements. Due to the analytically described potential function inside the local field, the resulting overall force using this approach is of the same order of convergence. Therefore, no loss of accuracy of the derived field quantities occurs. The convergence of the total forces, computed by the classical approach, indicates the expected linear behaviour. The accuracy of the computed values is influenced by the numerically-obtained derivatives. This shows that the results obtained by the classical method are inherently inaccurate when compared to the accuracy of the potential solution.

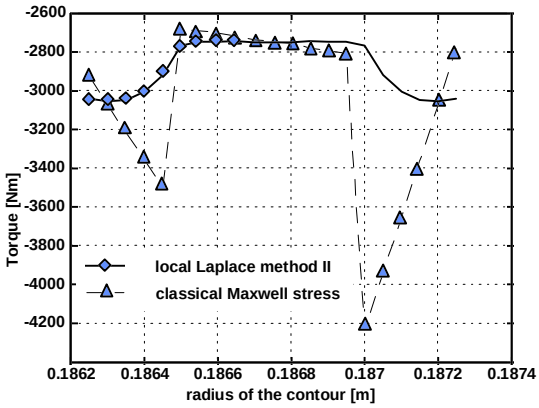


Fig. 18. Variation of the torque calculated along different contours inside the air gap.

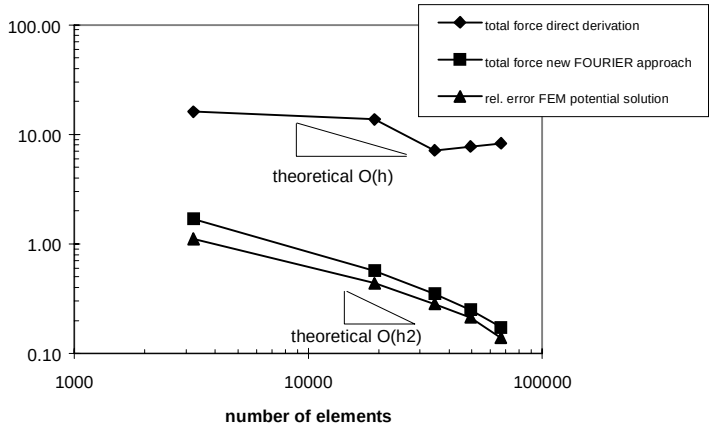


Fig. 19. Comparison of the convergence behaviour of the FEM potential solution with both the direct derivation and the derivative-free approach.

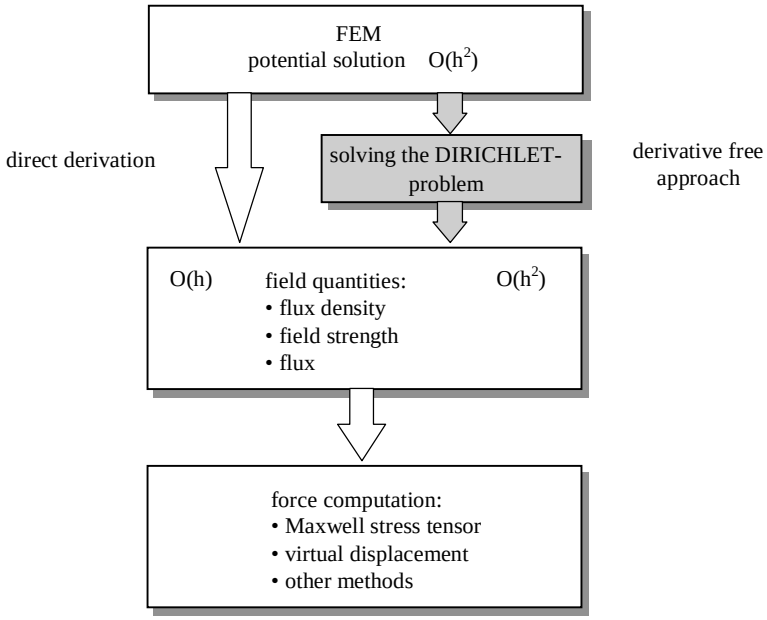


Fig. 20. Additional step during post-processing to enhance the accuracy of derived field quantities.

The use of the proposed approach to enhance the accuracy of computed field quantities starting from an existing potential solution demands an additional step during the post-processing of the FEM analysis (Fig. 20). Having obtained a FEM potential solution, the user only has to define the surface of integration Γ on which the field quantities or forces have to be calculated. Defining an arbitrary contour allows the computation of field quantities or forces along it as well.

5 Conclusions

The effective new design of modern and competitive electromagnetic devices, such as electrical machines, requires state of the art numerical simulation techniques.

Nowadays, field simulations are required considering specific coupled field phenomena. Therefore, particular attention must be paid to the numerical stability of the overall system of equations. If the structure of the overall coupled system is known, specific equation solvers are available to obtain the desired solution.

Numerical computational methods, such as the finite element method combined with numerical optimisation algorithms yield automated design procedures towards technical products having specific physical and/or economical properties.

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