Solution techniques for the hybrid systems arising from field-circuit coupled electromagnetic models

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Abstract

Electromagnetic field-circuit coupled models of induction machines give raise to linear systems with symmetric and indefinite system matrices. The sparse stiffness matrix of the finite element discretization is coupled to algebraic equations describing the external electric circuit connections. As solution techniques for the coupled systems, a variant of the Quasi Minimal Residual (QMR) method is compared to the Minimal Residual (MINRES) method. The experiment shows that QMR, which allows the use of an indefinite preconditioner, outperforms MINRES with a positive preconditioner. The latter solution strategies for the indefinite systems are compared to the Conjugate Gradient (CG) method applied to the positive definite Schur complements of the same systems. The different solution and preconditioning approaches are tuned to the particular electromagnetic application and are demonstrated by a realistic example.

1 Introduction

Expensive prototyping of electromagnetic devices in design and optimization procedures has largely been replaced by numerical simulation. Flexibility, reliability and speed are the major requirements for the simulation software. Such software is mostly based on the use of finite elements because of the complex geometries and non-linear material characteristics, typical for electrical power transducers. The working principle of a lot of electromagnetic devices relies on the interaction of an electric field with a magnetic field. Since both fields are linearly coupled by the Maxwell equations, the description of the coupled problem by one system matrix is particular attractive. In some cases, one of both fields is easily and to a sufficient accuracy described by a lumped parameter model, whereas the other field requires an entire 2D or 3D finite element discretization [1]. The differences in the natures and the discretizations of the fields yield a hybrid, symmetric and indefinite coupled system of equations. As the system has to be solved repeatedly in a transient simulation, it is important to pay attention to the particular properties of the system matrix and the choice of appropriate solvers and preconditioners. Indefinite systems of mixed formulations are also studied for models of other physical phenomena in [2] and [3].

2 Simulation of induction machines

The working principle of an induction motor is based on the interaction of an electric field and a magnetic field. Alternating currents in the three-phase stator windings of an induction machine excite a rotating magnetic field in the air gap of the machine (Fig. 1). If the angular velocity of the rotor differs from the one of the rotating field, currents are induced in the short cut rotor bars. The interaction of the stator field and the rotor field gives raise to an electromagnetic torque which drives the rotor [4].
The simulation of induction machines involves the computation of the electric and magnetic behaviour. Because of the time-dependent excitation, the magnetic saturation of the iron and the

motion of the rotor, a transient description is required. The magnetic flux distribution is strongly influenced by the presence of materials with a relative difference in magnetic permeability of a factor 1000. The electric currents, on the other hand, only exist in the conductors.
From the modelling point of view, a distinction is made between the solid rotor bars and stator windings. A solid conductor is a massive conducting piece of material. The voltage drop remains constant across the cross-section of the solid conductor. Because of the skin effect, however, the current density depends on the location in the conductor. A stranded conductor is the approximation for a winding. Due to the geometrical dimensions of the individual strands, skin effect is negligible. Therefore, the current density is constant along the cross-section of the conductor. The voltage drop is not the same for each wire.
The long and cylindrical geometry of the machine that we consider, enables the use of a 2D model. The magnetic flux density $\mathbf{B} = \left(B_x, B_y, 0\right)$ is then a vector in the 2D-plane whereas the electric voltage drop ∇V and the current density $\mathbf{J} = \left(0,0,J_z\right)$ are perpendicular to this plane. The resistances and inductances of the stator end-windings and the rotor end-ring have a strong impact on the behaviour of the machine [5]. Also, the machine may be operated by a non-sinusoidal source or invertor supply. These effects are considered as lumped parameters of an external electric circuit coupled to the 2D magnetic field model.

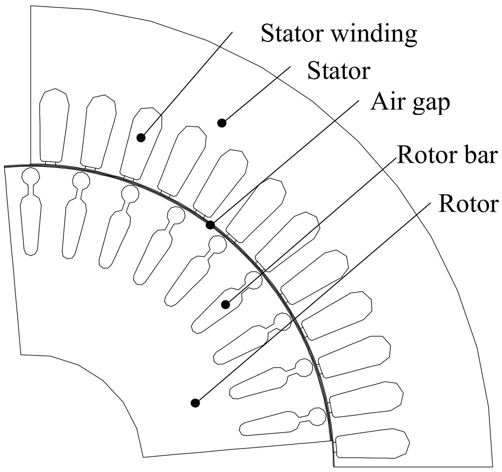

Fig. 1: Geometry of an induction motor.

3 Derivation of the mathematical and numerical model

3.1 Magnetic model

The wave length of the considered electromagnetic phenomena exceeds the dimensions of the applications. Based on this observation, one can simplify the Maxwell equations to obtain the following system of equations for magnetodynamic fields. A distinction is made between solid conductors and stranded conductors [6].

$$-\frac{\partial}{\partial x}\left(\nu\frac{\partial A_z}{\partial x}\right)-\frac{\partial}{\partial y}\left(\nu\frac{\partial A_z}{\partial y}\right)+\sigma\frac{\partial A_z}{\partial t}=\frac{\sigma}{\ell}V_{sol} \qquad \text{(solid conductors)}, \tag{1}$$

$$-\frac{\partial}{\partial x}\left(\nu\frac{\partial A_z}{\partial x}\right)-\frac{\partial}{\partial y}\left(\nu\frac{\partial A_z}{\partial y}\right)=\frac{N_t}{\Delta_{str}}I_{str} \qquad \text{(stranded conductors)}, \tag{2}$$

$$I_{sol} = G_{sol}V_{sol} - \int_{\Omega} \sigma \frac{\partial A_z}{\partial t} \,\mathrm{d}\,\Omega \qquad \text{(solid conductors),} \qquad (3)$$

$$V_{str} = R_{str}I_{str} + \frac{N_t \ell}{\Delta_{str}} \int_{\Omega} \frac{\partial A_z}{\partial t} \,\mathrm{d}\,\Omega \qquad \text{(stranded conductors).} \qquad (4)$$

A_z is the z-component of the magnetic vector potential $\mathbf{A}$ defined by the relation $\mathbf{B} = \nabla \times \mathbf{A}$ where $\mathbf{B}$ denotes the magnetic flux density. ν and σ are the reluctivity and the conductivity. ℓ is the length of the 2D model. A solid conductor is described by (1) and (3) as a function of the voltage V_{sol}, the current I_{sol} and the admittance G_{sol}. A stranded conductor with N_t turns and a cross-section Δ_{str}, is described by (2) and (4) as a function of the current I_{str}, the voltage V_{str} and the resistance R_{str}. Eq. (1) and (2) are partial differential equations that have to be solved on the domain. Eq. (3) and (4) are integral relations representing the coupling between the magnetic field and the electric circuit.

3.2 Discretization

Equations (1)-(4) are discretized in space using linear triangular finite elements. The time discretization is the Galerkin time-stepping scheme ($\alpha = 2/3$) with fixed time step Δt.

3.3 External circuit model

A lumped parameter model is described by the Kirchhoff Voltage Law, the Kirchhoff Current Law and a branch current-voltage relation for each branch. Writing all possible equations yields an overdetermined system of equations. Furthermore, solid conductors and capacitors are voltage driven branches. Their currents $\mathbf{i}_T$ are described by branch relations of the form

$$\mathbf{i}_T = \mathbf{G}_T \mathbf{v}_T + f_T(\mathbf{A}) \qquad (5)$$

in terms of the unknown voltages $\mathbf{v}_T$, the magnetic vector potentials $\mathbf{A}$ and the admittance matrix $\mathbf{G}_T$. Similarly, the voltages $\mathbf{v}_L$ across stranded conductors and inductors are described by

$$\mathbf{v}_L = \mathbf{R}_L \mathbf{i}_L + f_L(\mathbf{A}) \qquad (6)$$

in terms of unknown currents $\mathbf{i}_L$, the magnetic vector potentials $\mathbf{A}$ and the resistance matrix $\mathbf{R}_L$. The restrictions due to the hybrid nature of the circuit elements and their arbitrary interconnection are resolved by applying the graph theory to the circuit [7]. From here on, it is assumed that it is possible to put all voltage driven branches in a tree and all current driven branches in the associated co-tree. The more general case is pointed out in [7]. The topological treatment is represented by fundamental incidence matrices. The fundamental cutset matrix $\mathbf{D} = [\mathbf{1} \quad \mathbf{D}_L]$ represents the incidences of the circuit branches to the fundamental cutsets. The fundamental loop matrix $\mathbf{B} = [\mathbf{B}_T \quad \mathbf{1}]$ represents the incidences of the circuit branches to the fundamental loops [8].

An independent set of Kirchhoff Laws is obtained by considering the cutset equations

$$\mathbf{D}_L \mathbf{i}_L + \mathbf{i}_T = 0 \qquad (7)$$

and the loop equations

$$\mathbf{B}_T \mathbf{v}_T + \mathbf{v}_L = 0. \qquad (8)$$

3.4 Magnetic-electric field-circuit coupling

Substituting the discrete equivalents of (5) and (6) in (7) and (8), adding the discretized versions of (1)-(4) and scaling by $\chi = \frac{\alpha \Delta t}{\ell}$ leads finally to the coupled system

$$
\begin{bmatrix} \alpha\mathbf{K}+\dfrac{\mathbf{R}}{\Delta t} & \alpha\mathbf{Q}_T & -\alpha\mathbf{P}_L \\ \alpha\mathbf{Q}_T^T & \chi\mathbf{G}_T & \chi\mathbf{D}_L \\ -\alpha\mathbf{P}_L^T & -\chi\mathbf{B}_T & -\chi\mathbf{R}_L \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A} \\ \mathbf{v}_T \\ \mathbf{i}_L \end{bmatrix} = \mathbf{load}\,. \tag{9}
$$

$\mathbf{K}$, $\mathbf{R}$, $\mathbf{Q}_T$ and $\mathbf{P}_L$ follow from discretizing (1), (2), (3) and (4) [7]. The vector **load** depends on the known voltage and current sources and the solution at the previous time step [7].

4 Solution of the matrix problem

4.1 Properties

The matrix is symmetric thanks to the graph property $\mathbf{D}_L = -\mathbf{B}_T^T$ [8] and the symmetrization by χ. The finite element matrix block $\alpha\mathbf{K}+\mathbf{R}/\Delta t$ arises from a finite element discretization of the parabolic and elliptic partial differential equations (1) and (2) and is positive definite. $\mathbf{G}_T$ and $\mathbf{R}_L$ are positive diagonals because of the assumption made in Section 3.3. In the general case, $\mathbf{G}_T$ and $\mathbf{R}_L$ are symmetric positive definite matrices. The cutset equations preserve the positive definiteness of the FEM matrix part. The loop equations are responsible for a negative definite diagonal block. The cutset equations correspond to the unknown voltages and are directly related to the magnetic vector potentials by the law of Faraday-Lenz. The loop equations, corresponding to the unknown currents, introduce a duality with respect to the magnetic vector potentials. The indefiniteness of the coupled system is related to this physical duality. An appropriate congruence transform and Sylvester's law of inertia revail that the number of negative eigenvalues equals the number of loop equations. The benchmark model in Fig. 2 has one cutset equation and one loop equation. The matrix may be ill-scaled due to the relative differences between the diagonal blocks in the mixed formulation.

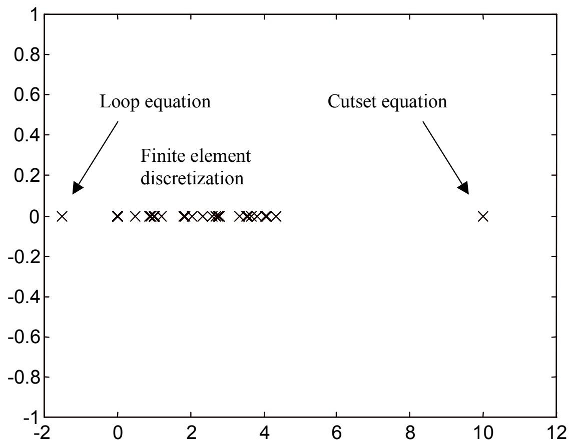

Fig. 2: Spectrum of a benchmark system matrix.

4.2 Solving the indefinite system

Preconditioned Krylov subspace solvers are very effective for large and sparse systems of equations [9]. Appropriate algorithms for symmetric, indefinite systems are the Minimal Residual (MINRES) and the Symmetric LQ (SYMMLQ) methods [10]. Both methods are based on the Lanczos process for symmetric matrices. The coefficient matrix of the preconditioned system has to be symmetric. Furthermore, the minimization in MINRES is performed using the norm associated with the inner product to which the coefficient matrix is symmetric. This is only possible if the preconditioner defines a norm and is thus symmetric positive definite.

A better convergence of the Krylov subspace method is expected when both the positive and the negative eigenvalues, are approximated in the preconditioner. To benefit from an indefinite preconditioner, MINRES is replaced by a variant of the Quasi-Minimal Residual (QMR) method for symmetric, indefinite systems [11]. The Lanczos bi-orthogonalization process in QMR simplifies in the case of a symmetric preconditioner. The true minimization of MINRES is replaced by a quasi-minimization. The resulting algorithm requires essentially the same amount of work and storage as MINRES. This approach enables the application of either which symmetric preconditioner to the indefinite system.

4.3 Positive definite alternative

The negative eigenvalues can be avoided by using voltage unknowns only. The Schur complement of the loop equations is positive definite and corresponds to a full nodal analysis [1]. Here, the Schur complement of the entire circuit part is applied. Consider the partitioning of the indefinite system (9):

$$\begin{bmatrix} \alpha\mathbf{K} + \dfrac{\mathbf{R}}{\Delta t} & \alpha\mathbf{T} \\ \alpha\mathbf{T}^T & \mathbf{C} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A} \\ \mathbf{U} \end{bmatrix} = \mathbf{load}\,. \tag{10}$$

The Schur complement

$$\mathbf{S} = \alpha\mathbf{K} + \frac{\mathbf{R}}{\Delta t} - \frac{\alpha^2}{\chi}\mathbf{T}\mathbf{C}^{-1}\mathbf{T}^T\,, \tag{11}$$

is positive definite. The elimination of the circuit unknowns cause a significant fill-in of the finite element matrix part. For medium sized problem, already, the memory resources are unsufficient and the matrix-vector product in the Krylov subspace solver becomes unacceptable expensive. The matrix-vector product can be performed implicitly using a factorization of $\mathbf{C}$. The system is solved by the Conjugate Gradient (CG) method. The lack of a good preconditioner for $\mathbf{S}$ is a disadvantages of this approach. It is possible to apply a good preconditioner for $\alpha\mathbf{K} + \mathbf{R}/\Delta t$ to the Schur complement as well. In the numerical experiments, an Algebraic Multigrid (AMG) [12] defined for $\alpha\mathbf{K} + \mathbf{R}/\Delta t$, is used as a preconditioner for $\mathbf{S}$.

5 Example

The induction motor of Fig. 1 is a four-pole 45 kW motor with as rated values for the efficiency, the speed, the voltage and the current, 93.5%, 1470 rpm, 660 V and 49.5 A respectively. A transient simulation has to reflect results of the stationary behaviour of the machine. The fundamental frequency and higher harmonics due to saturation and slotting are of interest. The relative motion of the rotor and stator slots determines the time-step. Two periods are simulated with 2048 time steps in total. The geometry is discretized by 6010 first-order finite elements. The topological treatment of the external electric circuit yields 354 circuit equations considered in the entire system matrix. The magnetic flux lines for one position and time instant are drawn in Fig. 3.

Fig. 4 shows the convergence of QMR and GMRES with a Jacobi preconditioner compared to MINRES with the Jacobi preconditioner of which all negative diagonal elements are made positive. The effect of the indefinite preconditioner used with QMR outstands the true minimization of MINRES. In Fig. 5 the convergence of CG applied to the Schur complement is shown for several choices of preconditioners.

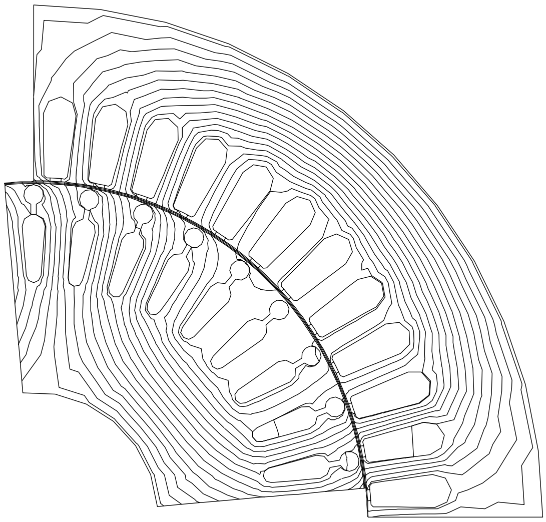

Fig. 3: Magnetic flux lines of the induction motor.

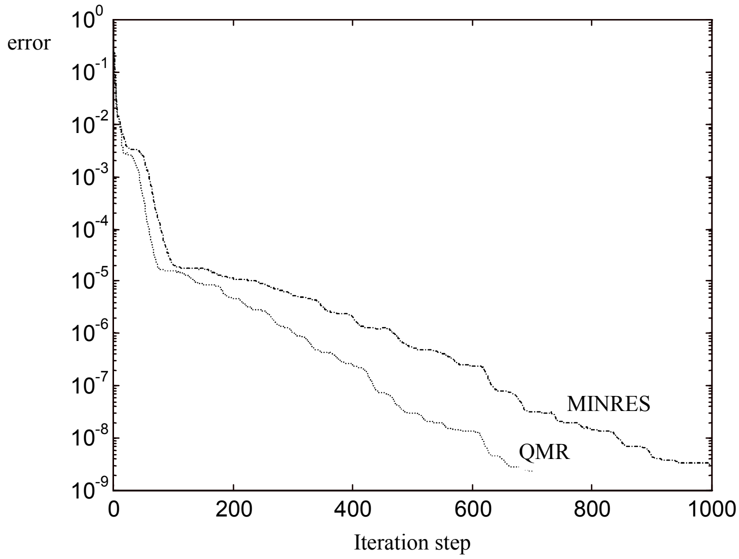

Fig. 4: Convergence of QMR compared to MINRES.

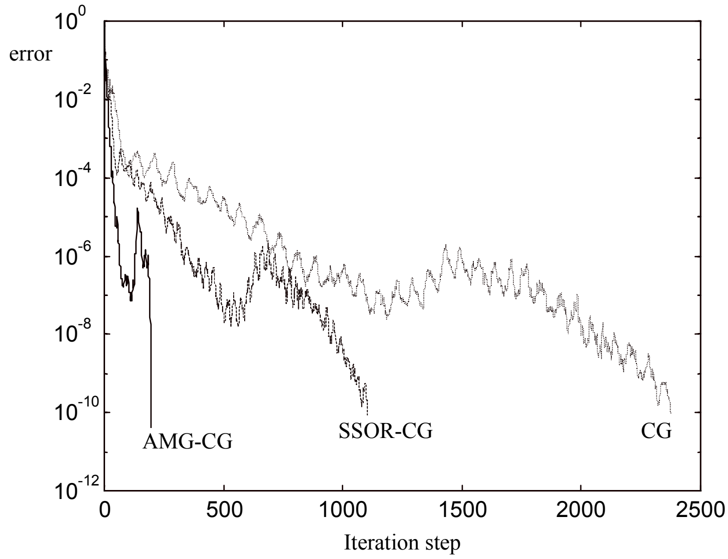

Fig. 5: Convergence of CG applied to the Schur complement for different preconditioners.

6 Conclusions

The finite element discretization of a coupled magnetic-electric field-circuit transient simulation yields a symmetric indefinite system matrix. The Quasi-Minimal Residual method with a preconditioner considering the few negative eigenvalues, performs better than the Minimal Residual approach. An alternative is implicitly solving the positive definite Schur complement by the Conjugate Gradient method. Numerical experiments are performed on an induction motor model.

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