Weak Coupling of Magnetic and Vibrational Analysis Using Local Forces

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Abstract: A weak coupling between magnetostatic and elasticity equations is derived from energy considerations. The coupling term results directly into a finite element expression for the nodal electromagnetic forces, which can be used as source terms for an elasticity or vibration analysis. The relative contribution of the stator's modal shapes in the deformation excited by this force distribution is calculated. As an example, the coupling is used to analyse the vibrational behaviour of a 6/4 switched reluctance machine.

Keywords: finite element methods, electromagnetic forces, modal analysis, mechanical factors, coupled problems.

I. INTRODUCTION

The electromagnetic field inside an electrical machine and its mechanical structure will determine the machine's behaviour in producing vibrations and noise. The link between the magnetic and the mechanical analysis is the electromagnetical force exerted by the magnetic field on stator and rotor. To take stator deformations into account, a local force formulation is needed. A finite element based expression for local electromagnetic forces is presented. In the finite element analysis, forces can be calculated at every node of the mesh and this force distribution can be used as an input (source terms) to the subsequent mechanical analysis. From the modal shapes of the stator and the force distribution, mode participation factors can be determined (as a function of rotor position), indicating the relative importance of the modal shapes towards the machine's vibrations and noise. This analysis is illustrated by example of a 6/4 switched reluctance machine (SRM).

II. THE MAGNETO-MECHANICAL SYSTEM

Both magnetostatic and elasticity finite element methods are based upon the minimisation of an energy function. The *elastic energy* stored in a body with deformation a ($x_i=x_{i,0}+u_i$, $y_i=y_{i,0}+v_i$, $a_i=[u_i\ v_i]^T$) is [1]

$$U = \frac{1}{2} a^T K a \tag{1}$$

where K is the mechanical stiffness matrix, determined by geometry and material properties ρ, E and ν, i.e. density, Young modulus and Poisson modulus. The *magnetic energy* stored in an unsaturated system with magnetic vector potential A is [2]

$$W = \frac{1}{2} A^T M A \tag{2}$$

where M is the system's magnetic 'stiffness' matrix, determined by its geometry and the magnetic permeability μ. Considering the similar form of (1) and (2), it is investigated whether the following system of equations can support a coupled magneto-mechanical analysis:

$$\begin{bmatrix} M & D \\ C & K \end{bmatrix} \begin{bmatrix} A \\ a \end{bmatrix} = \begin{bmatrix} T \\ R \end{bmatrix} \tag{3}$$

where T and R are the magnetic and mechanical source term vectors respectively. R represents forces other than those of electromagnetic origin. The coupling matrices C and D can be evaluated considering the total energy in the system:

$$E = U + W = \frac{1}{2} a^T K a + \frac{1}{2} A^T M A . \tag{4}$$

The partial derivatives of E with respect to the unknowns $[A\ \ a]^T$ give the equations of the total system (3) with T=0, R=0:

$$\frac{\partial E}{\partial A} = M A + \frac{1}{2} a^T \frac{\partial K(A)}{\partial A} a = 0 , \tag{5}$$

$$\frac{\partial E}{\partial a} = K a + \frac{1}{2} A^T \frac{\partial M(a)}{\partial a} A = 0 . \tag{6}$$

The coupling terms C and D are thus given by

$$D = \frac{1}{2} a^T \frac{\partial K(A)}{\partial A} \tag{7}$$

$$C = \frac{1}{2} A^T \frac{\partial M(a)}{\partial a} \tag{8}$$

III. ELECTROMAGNETIC FORCES

Using the coupling terms C and D, it is possible to solve the matrix system (3) directly. Solving this strongly coupled system requires an iterative solver that can handle a non-sparse asymmetrical system, e.g. GMRES. Therefore it is useful to examine the weakly coupled version of (3), since this will lead to an expression for the electromagnetic nodal

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forces, and the equation solvers for sparse symmetric systems can still be used. Rearranging the second equation in (6) to

$$K a = -\frac{1}{2} A^T \frac{\partial M(a)}{\partial a} A = F_{em} \tag{9}$$

reveals a means to calculate the nodal electromagnetic forces F_{em} from magnetic vector potential A and the partial derivative of the magnetic stiffness matrix M with respect to deformation a. This expression for F_{em} can also be found directly by deriving magnetic energy W with respect to displacement:

$$F_{em} = -\frac{\partial W}{\partial a} = -\frac{\partial}{\partial a}\left[\frac{1}{2} A^T M A\right] \tag{10}$$

where the unknowns A have to be considered constant. If it is assumed that the material properties E, ν and ρ do not depend on the vector potential A (e.g. neglecting magnetostriction), then the coupling term D vanishes. The system (3) is uncoupled into

$$\begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}\begin{bmatrix} A \\ a \end{bmatrix} = \begin{bmatrix} T \\ R + F_{em} \end{bmatrix} \tag{11}$$

since F_{em} replaces the coupling term C. Equation (11) can be solved using a simple cascade procedure and its validity is tested against analytical models, as presented in the full paper.

The derivation $\partial M/\partial a$ is illustrated briefly in the full paper for the case of first order triangular elements. In this abstract, suffice it to give the results. For the magnetic element matrix

$$M_{ij}^e = \frac{1}{4\mu\Delta}\left[b_i b_j + c_i c_j\right] \tag{12}$$

with permeability μ, element area Δ and the familiar shape function coefficients $a_1=x_2y_3-x_3y_2$, $b_1=y_2-y_3$, $c_1=x_3-x_2$, the partial derivative with respect to u_1 is

$$\frac{\partial M_{ij}^e}{\partial u_1} = \frac{1}{4\mu\Delta}\begin{bmatrix} 0 & c_1 & -c_1 \\ c_1 & 2c_2 & c_3 - c_2 \\ -c_1 & c_3 - c_2 & -2c_3 \end{bmatrix} - \frac{2b_1}{\Delta} M_{ij}^e . \tag{13}$$

Rather than calculating the energy difference between two finite element solutions, the partial derivative represents more accurately the essence of virtual work [3]. There is no need for a second magnetic finite element solution and no numerical derivations are performed.

IV. EXAMPLE: 6/4 SRM

A. Nodal Forces

The geometry of the 6/4 SRM is shown in Fig.1a. The coil system indicated is current excited and generates the magnetic field shown in Fig.1b. This magnetic field is used to evaluate the local electromagnetic forces F_{em} given in Fig.2 for the stator only. For this rotor position, the machine produces torque due to the (net) forces alongside the stator teeth. The forces pointing inwards are larger than the torque generating forces, but only cause the stator to deform.

B. Mode Shapes and Modal Participation

Using the stator's mechanical matrices K (stiffness), M_m (mass) and C_m (damping), the eigenvectors and eigenvalues of the mechanical structure can be found. These constitute the stator deformation mode shapes, some of which are shown in Fig.3. Table 1 lists the first 20 modes with their frequencies and their (normalised) participation factor for the force distribution under consideration. The mode participation factor can be determined using [4]

$$\Gamma_i = \frac{1}{m_i \phi_i^2} \sum_j \phi_i(x_j)^T p(x_j) \tag{14}$$

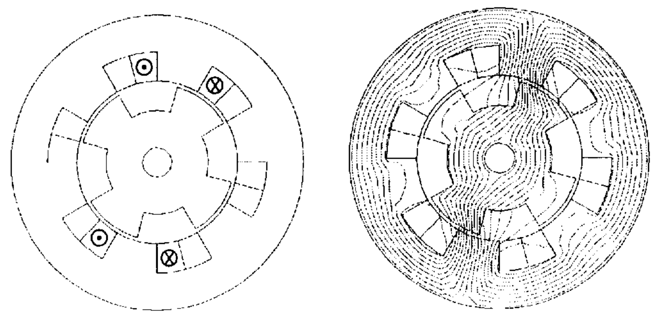

Fig. 1. a) Geometry of the 6/4 SRM and b) equipotential lines for excitation according to a).

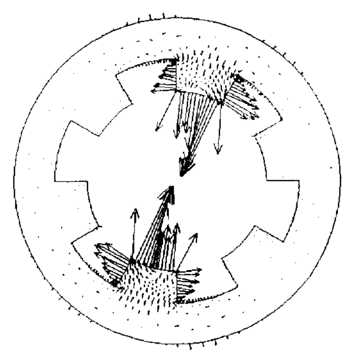

Fig. 2. Nodal force distribution calculated from the magnetic field.

where $\phi_i(x_j)$ is the displacement of the i^{th} mode shape at the j^{th} node, $p(x_j)$ is the force at the j^{th} node and m_i is the generalised mode mass.

In Table 1, several modes are paired because they represent deformation patterns that differ an angular shift only, e.g. modes (1,2) are shifted over 90°, modes (4,5) over 45° and modes (9,10) over 22.5°.

From Table 1 it is seen that the 8^{th} mode shape, the uniform shrinking and expanding of the stator structure, has the largest contribution and determines the greater part of the vibrational behaviour of the machine. The squaring modes (9,10) have a substantial contribution and also the ovalization modes (4,5) are clearly present. The triangular modes 6 and 7 have no significant contribution since there is no triangular symmetry in the forces. Note that the contribution of the 3^{rd} mode shape, the rigid body rotation, is a measure for the torque of the SRM: this value can be compared for different rotor positions to indicate the machine's torque efficiency and its susceptibility to torque ripple.

Table 1. Participation factor of modal shapes in force distribution

mode number	frequency (Hz)	mode participation factor (normalised)
1, 2	rigid translations	0.0270
3	rigid rotation	0.0227
4, 5	334.9	**0.4676**
6	783.8	0.0269
7	1028.1	0.0005
8	1311.3	**0.5810**
9, 10	1598.5	**0.4941**
11, 12	1862.3	0.1286
13, 14	2488.7	0.0671
15	2851.3	0.0211
16, 17	3249.2	**0.3114**
18	4019.5	0.1209
19, 20	4530.6	**0.2542**

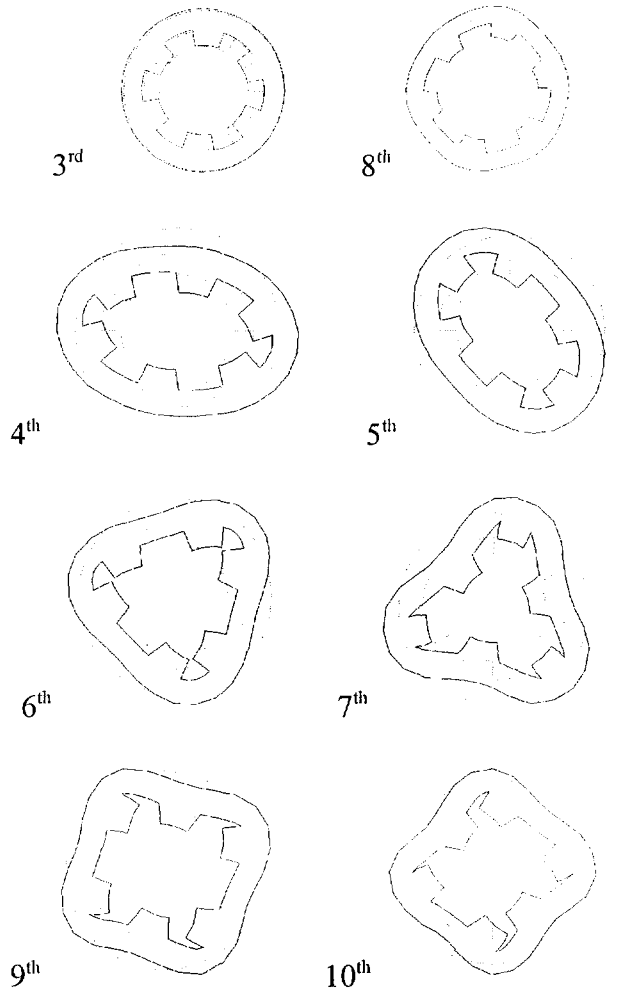

Fig. 3. Selected modal shapes for the 6/4 SMR stator structure. The mode numbers are assigned according to ascending frequency.

The force distribution in Fig.2 is only a snapshot: a full modal analysis has to consider the force distribution for different rotor positions, corresponding to different time steps and coil excitations. When the participation factors Γ_i are known for different rotor positions (and time instants), the set of equations

$$\ddot{q}_i + 2\zeta_i\omega_i\dot{q}_i + \omega_i^2 q_i = \Gamma_i(t) \tag{15}$$

can be solved for all significant modes ϕ_i, giving their generalised co-ordinates q_i as a function of time (ω_i = eigenfrequency, ζ_i = modal damping factor). When the mechanical damping C_m is assumed to be proportional, the system of equations (15) can be uncoupled and solved separately [4].

Note that solving the set of equations (15) in the time domain, eliminates inaccurate assumptions on the frequency behaviour of the local electromagnetic forces.

V. CONCLUSION

A weak coupling between magnetic and mechanical analysis is derived, leading to a finite element based expression for the nodal electromagnetic forces. The modal shapes and their participation factor can be calculated for different rotor positions. From these values, stator resonances and the noise frequency spectrum can be anticipated at the design level.

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