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An Algebraic Multigrid Method for Solving Very Large Electromagnetic Systems

Ronny Mertens, Herbert De Gersem, Kay Hameyer and Ronnie Belmans
KATHOLIEKE UNIVERSITEIT LEUVEN, DEP. EE (ESAT), DIV. ELEN
Kardinaal Mercierlaan 94
B-3001 Leuven, BELGIUM

Domenico Lahaye, Stefan Vandewalle and Dirk Roose
KATHOLIEKE UNIVERSITEIT LEUVEN, DEP. COMPUTER SCIENCE
Celestijnenlaan 200A
B-3001 Leuven, BELGIUM

Abstract - **Although most finite element programs have very effective iterative solvers such as ICCG and SSORCG, the solution time becomes long for very large systems. Convergence and thus the total solution time can be improved by using better preconditioners such as multigrid methods. Algebraic multigrid methods have the supplementary advantage that no geometric information is needed. The implemented algebraic multigrid method reduces the overall computation time by a factor of 6 compared to a SSORCG solver.**

INTRODUCTION

In finite element programs, direct methods are nowadays often replaced by iterative methods to solve the system of discretised linear equations. Stationary methods such as Jacobi, Gauss-Seidel and successive overrelaxation are straightforward to implement but usually not very effective. The conjugate gradient method (CG), a non-stationary method, is harder to apply, but very effective when used in combination with a good preconditioner. Symmetric successive overrelaxation (SSOR) and the incomplete Cholesky decomposition (IC) are typically used as preconditioner for the CG method. Multigrid methods can also be used as preconditioners or as solvers to obtain even more efficient iterative methods.

ALGEBRAIC MULTIGRID METHOD

Basic iterative schemes for solving the matrix equation

$$\boldsymbol{A}\,\boldsymbol{x} - \boldsymbol{f} = 0 \tag{1}$$

often reduce the high frequency components of the error vector $\boldsymbol{e}$ very effectively (fig. 1.i).

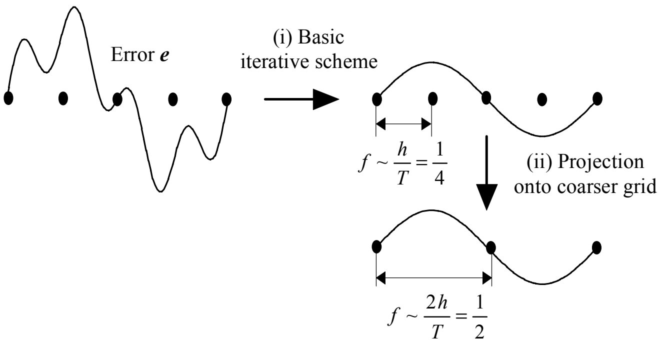
Error e
(i) Basic iterative scheme
$f \sim \frac{h}{T} = \frac{1}{4}$
(ii) Projection onto coarser grid
$f \sim \frac{2h}{T} = \frac{1}{2}$

Fig. 1: Evolution of the error $\boldsymbol{e}$

Convergence stalls, however, as soon as the error becomes smooth. Smooth errors look more oscillatory when projected onto coarser grids (fig. 1.ii). This is exploited in multigrid methods, which involve a hierarchy of continuously coarser meshes [3]. The error on the fine grid is smoothed by applying a few steps of a basic iterative scheme. This smooth error is projected onto a coarser mesh, where a correction term is computed. The correction term is interpolated back to the fine grid and added to the existing approximation. This method is applied recursively to compute the coarser grid corrections until the cost of computing this correction by a direct solver becomes negligible.

Algebraic Multigrid Method as Solver

Standard multigrid methods require a hierarchy of meshes constructed by using the geometry of the problem. This makes the implementation of these methods more complex compared to a CG method. Algebraic multigrid methods (AMG) automatically construct a sequence of coarser grids using information about the matrix $\boldsymbol{A}$ only. This makes AMG attractive as 'black box' solver [2,3]. Once the set of coarse grids is defined, AMG computes each of the consecutive approximations to (1) by going back and forward between the finest and the coarsest grid.

Algebraic Multigrid Method as Preconditioner

The general form of a stationary iterative method is

$$\boldsymbol{x}^{k+1} = \boldsymbol{x}^{k} - \boldsymbol{M}^{-1}\left(\boldsymbol{A}\boldsymbol{x}^{k} - \boldsymbol{f}\right) \tag{2}$$

where $\boldsymbol{M}$ denotes the preconditioner associated with the method. If the residual vector $\boldsymbol{r}^{k} = \boldsymbol{A}\boldsymbol{x}^{k} - \boldsymbol{f}$ is taken as the right hand side, zero as initial solution and only 1 iteration step is performed, $\boldsymbol{M}^{-1}$ is applied to $\boldsymbol{r}^{k}$, which is exactly what is needed to use AMG as preconditioner for the CG method (AMGCG) [1].

Comparison of the Different Solvers

To compare the total solution time of different solvers, a synchronous line-start motor excited with permanent magnets is taken as an example. Saturation plays an important role in the behaviour of this machine. Fig. 2 shows the initial mesh and fig. 3 the field plot of an intermediate adaptation step.

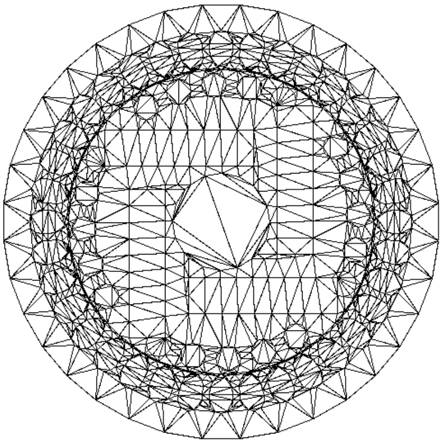

Fig. 2: Initial mesh of a synchronous line-start motor (1092 nodes)

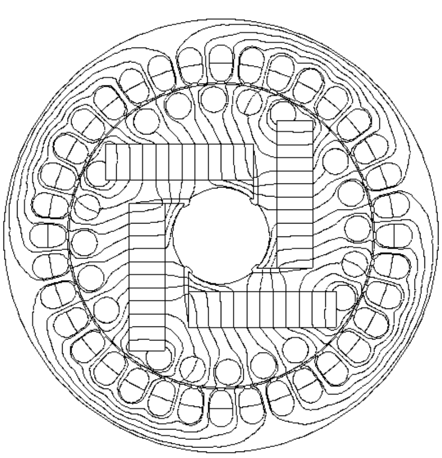

Fig. 3: Field plot of a synchronous line-start motor (11164 nodes)

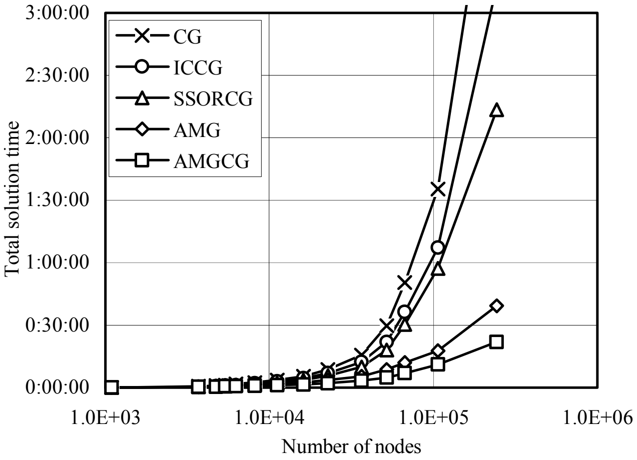

Fig. 4: Total solution times after each adaptation step

Values of the nodal flux densities weighted with the energy in an element are used as an a posteriori error estimator. 13 adaptation steps were calculated on a HP C-160 workstation. Fig. 4 shows the total solution times after each adaptation step for the different methods. To solve the system after 13 adaptation steps (243572 unknowns), AMGCG required only 23 CG steps in the last Newton step, while SSORCG and ICCG needed respectively 2438 and 2467 CG steps.

Conclusion

Despite the extra cost of applying an algebraic multigrid method as preconditioner for the conjugate gradient method (AMGCG), the solution time is reduced by a factor of 2 for smaller problems and a factor of 6 for very large problems compared to symmetric successive overrelaxation as preconditioner (SSORCG) . The increase of 50 % (41 MB instead of 29 MB for 100000 nodes) in memory requirements for storing and solving the system of linear equations is therefore worth paying. As AMG can be used as a 'black box' solver, AMGCG is very well suited to solve 3D problems where solution times increase even more rapidly and the geometric construction of coarse meshes is even more problematic than in the 2D case.

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