Higher accuracy of 3D finite element solutions for field and force computations

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Abstract— **A new method for the accurate computation of the field quantities and forces in three-dimensional finite element models is presented. Solving a local Dirichlet problem enhances the accuracy of derived field quantities using already computed potentials. Derivatives and thus the values of flux density are calculated analytically in order to improve their order of convergence. A Fourier series is used to represent the local field solution of a three-dimensional electromagnetic problem. Test examples demonstrate the behaviour of the proposed method.**

I. Introduction

The conversion of useful energy in electromagnetic devices takes place in the air gap only. Numerical field computation techniques as the finite element method (FEM) are in common use to evaluate the essential field quantities and generated forces. To predict the forces and thus the behaviour of the electromagnetic device the highest possible accuracy of the solution is required. Particular attention must be paid to the computation of the air gap quantities of the flux density distribution and magnetic field strength. The mentioned quantities are derivatives of a potential formulation. The difficulty is found in the fact that the FEM approximates an arbitrary potential instead of obtaining the exact solution of a problem [1]. Noting this and assuming a small value of h as the maximum characteristic diameter of a finite element, the FEM is convergent of order $q+1$. The constant q describes the polynomial order of the elements used. With ε as the global error, the order of convergence for the potential solution is

$$\|\varepsilon\| \le C \cdot h^{q+1}. \qquad (1)$$

The factor C is independent from the size h of the elements and depends only on the

- type of discretization
- choice of shape function
- smoothness of the exact solution

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Equation (1) identifies the convergence problem transferred into an approximation problem. Using linear shape functions of first order the rate of convergence will be $O(h^2)$. Deriving the field quantities from the potential formulation numerically results in a rate of convergence $O(h)$ for those quantities i.e. when compared to the potential solution a loss in accuracy of order one. By using those field quantities e.g. for force calculations, this inherent inaccuracy consequently influences the results. This fact identifies the difficulty obtaining accurate field quantities as a problem of the order of convergence of the numerical method used.

By using an adaptively h-refined FEM discretization, the size of h varies from element to element. In this case, the order of convergence can be expressed by the degree of freedom (DOF) of the finite element mesh i.e. $O(h^2) = O(DOF^{-1/2})$.

To surmount the loss in accuracy shape functions of higher order can be used. On the other hand, this would result in fast increasing computational expenses. To avoid a large growth in computation time a good trade-off in accuracy and computation time is the use of linear shape functions and a local solution obtained by an analytical formulation a potential interpolating function and thus analytically derived field quantities of the same order of accuracy. When analysing electromagnetic devices mainly local values of the field quantities and the forces are of interest, so the proposed local solutions does not restrict the procedure of analysis.

Starting from the idea to approximate the local two dimensional field problem inside a circle by given potentials as the boundary values on its circumference analytically [2]-[4] to three-dimensional fields, the local field is computed inside a spherical volume. Again a Fourier series is used to obtain an analytical representation of the derivatives inside the sphere to avoid the loss in order of convergence and accuracy. The beforehand computed potential values using the FEM are assumed to be the boundary values of the local field problem and they are equally distributed on the surface of the spherical volume to compute the field values in the centre of the sphere analytically.

Using an existing FEM potential solution the proposed method describes a post-process operator practically applied to the air gap region of an electromagnetic device. No restrictions concerning the finite element discretization are assumed. The proposed method is independent of the finite element mesh inside the domain and the results obtained are converging against the values obtained by the classical evaluation of the potential solution using numerical derivatives if a very rough discretization or not suitable

parameters e.g. the number of boundary potential values or the radius of the spherical volume are chosen. The dependence of accuracy of the enhanced field solution on the necessary parameters to define the local field problem is discussed and the suitability of the proposed method are demonstrated by a test example.

II. Local 3D Field Solution

For a more accurate force calculation the aim is to improve the results of an existing field solution by a local post-process. The idea is to solve the three-dimensional Laplace equation for the magnetic scalar potential u

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \tag{2}$$

in source free and homogenous areas i.e. in the air gap of an electromagnetic device starting with an existing potential solution u. The local field problem is now defined by the known potential values equally distributed at the surface of a sphere assumed to be the boundary potential values of the local field problem. According to the co-ordinate transformation

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{3}$$

a spherical co-ordinate system is applied (Fig 1).

Using the Laplace equation (2) with the transformation (3) yields

$$\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}\right] = 0 . \tag{4}$$

Applying $u(r,\theta,\phi) = R(r)\cdot\Theta(\theta)\cdot\Phi(\phi)$ to (4) and the separation of the variables r, θ and ϕ, a general form of the functions $R(r)$, $\Theta(\theta)$ and $\Phi(\phi)$ depending on the potential u can be rewritten. Every solution of the Laplace equation (4), being finite for all θ, is a solution of

$$u_{m,n}(r,\theta,\phi) = \left(a r^n + b r^{-(n+1)}\right) P_n^m(\cos\theta)\cdot\left(\alpha\cos m\phi + \beta\sin m\phi\right) \tag{5}$$

where $m=0(1)\infty$, $n = m(1)\infty$, a, b, α and β are constants. P_n^m is the associated Legendre polynomial of the first kind. To simplify notation the surface harmonics

$$\begin{aligned} c_{m,n} &= P_n^m(\cos\theta)\cdot\cos m\phi \\ s_{m,n} &= P_n^m(\cos\theta)\cdot\sin m\phi \end{aligned} \tag{6}$$

are introduced. Assuming (5) to be a linear form, the potential in the origin is finite. The constants $\alpha_{m,n}(r)$ and $\beta_{m,n}(r)$ are linear combinations of r^n and $r^{-(n+1)}$. The summation

$$u = f(\theta,\phi) = \sum_{m=0}^{\infty}\sum_{n=m}^{\infty}\left[p_{m,n}\cdot c_{m,n}(\theta,\phi) + q_{m,n}\cdot s_{m,n}(\theta,\phi)\right]\cdot r^n \tag{7}$$

is a solution of (4). Here, the magnetic scalar potential u is completely described by the constants $p_{m,n}$ and $q_{m,n}$. It was the aim to calculate the magnetic flux density in a point using the values of known scalar potential in its vicinity. Consequently a spherical volume with known boundary potentials on its surface around this field point is chosen to determine the field quantity. The known boundary potentials are resulting from a beforehand executed FEM computation and determine all constants in (7). To calculate the magnetic field quantities in the centre of the spherical volume the Laplace equation has to be solved locally and spherically around this field point with radius r=R. The boundary potential values are available only as single values on the surface of the sphere. To distribute them equally on this surface the sphere co-ordinates ϕ are divided into J and θ into K equal angles $\Delta\phi$ and $\Delta\theta$ respectively.

$$\begin{aligned} 2\pi &= \sum_{j=1}^{J}\Delta\phi \\ \pi &= \sum_{k=1}^{K}\Delta\theta \end{aligned} \tag{8}$$

To satisfy (7) accurately the number of J and K must be large. On the other hand, large numbers will increase the computational expenses rapidly. With respect to the computation time and accuracy a good compromise has to be found. Practical values for J and K are given in a following section.

Assuming that ϕ and θ are the co-ordinates in the local system with the interesting field point in the centre of the

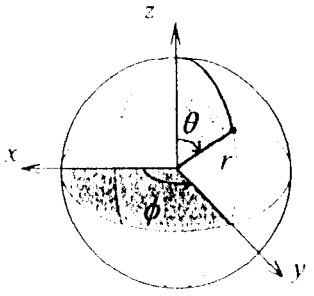


Fig. 1 Spherical co-ordinate system

sphere the coefficients $p_{m,n}$ and $q_{m,n}$ can be determined by a Legendre decomposition using the boundary potential values.

$$p_{m,n}\big|_{m=0} = \frac{\pi}{J\cdot K}\frac{1}{2R^n}\frac{(2n+1)(n-m)!}{(n+m)!}\sum_{k=1}^{K}\left[\sum_{j=1}^{J} f(\theta_k,\phi_j)\right] \cdot P_n^m(\cos\theta_k)\cdot\sin\theta_k$$

$$p_{m,n}\big|_{m>0} = \frac{\pi}{J\cdot K}\frac{1}{R^n}\frac{(2n+1)(n-m)!}{(n+m)!}\sum_{k=1}^{K}\left[\sum_{j=1}^{J} f(\theta_k,\phi_j)\cos m\phi_j\right] \cdot P_n^m(\cos\theta_k)\cdot\sin\theta_k$$

$$q_{m,n} = \frac{\pi}{J\cdot K}\frac{1}{R^n}\frac{(2n+1)(n-m)!}{(n+m)!}\sum_{k=1}^{K}\left[\sum_{j=1}^{J} f(\theta_k,\phi_j)\sin m\phi_j\right] \cdot P_n^m(\cos\theta_k)\cdot\sin\theta_k \tag{9}$$

Retaining the local co-ordinate system in (x', y', z'), the magnetic flux density in the original global co-ordinate system is

$$\mathbf{B} = \mu_0\mathbf{H} = -\mu_0\,\mathrm{grad}\,u \tag{10a}$$

or

$$(B_x, B_y, B_z) = -\mu_0\left(\frac{\partial u}{\partial x'}, \frac{\partial u}{\partial y'}, \frac{\partial u}{\partial z'}\right) \tag{10b}$$

Calculating the derivatives in the origin of the local co-ordinate system (Fig. 1) using $\theta = \pi/2$ and $\phi = 0$ in (7) it holds that

$$-\mu_0\left.\frac{\partial u}{\partial x'}\right|_{(0,0,0)} = -\mu_0\left.\frac{\partial u}{\partial r}\right|_{(0,0,0)} \tag{11}$$

Analogous to (11) the derivatives, in y' choosing $\theta = \phi = \pi/2$ and in z' respectively $\theta = 0$ and $\phi = \pi/2$, can be evaluated using (7). With respect to (11), applying (10b) to (7) by using (9) and with the Legendre terms

$$P_1^0(\cos\theta) = \cos\theta$$
$$P_1^1(\cos\theta) = \sin\theta \tag{12}$$

the components of the flux density in the centre of a sphere are explicitly rewritten by

$$B_x = -\mu_0\frac{3}{2}\frac{\pi}{J\cdot K\cdot R}\left[\sum_{k=1}^{K}\left(\sum_{j=1}^{J} f(\theta_k,\phi_j)\cos\phi_j\right)\cdot\sin^2\theta_k\right]$$

$$B_y = -\mu_0\frac{3}{2}\frac{\pi}{J\cdot K\cdot R}\left[\sum_{k=1}^{K}\left(\sum_{j=1}^{J} f(\theta_k,\phi_j)\sin\phi_j\right)\cdot\sin^2\theta_k\right] \tag{13}$$

$$B_z = -\mu_0\frac{3}{2}\frac{\pi}{J\cdot K\cdot R}\left[\sum_{k=1}^{K}\left(\sum_{j=1}^{J} f(\theta_k,\phi_j)\right)\cdot\cos\theta_k\cdot\sin\theta_k\right]$$

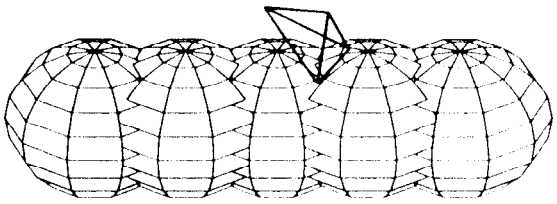
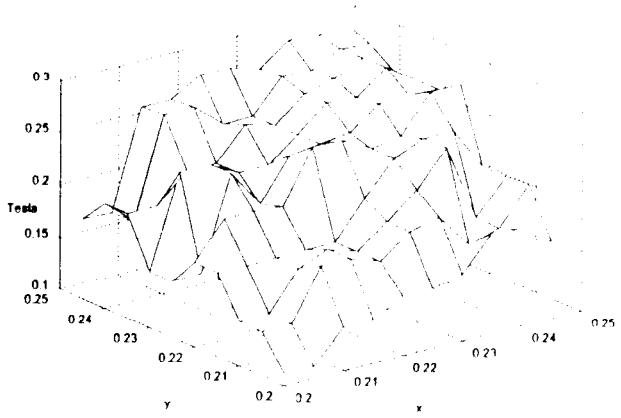
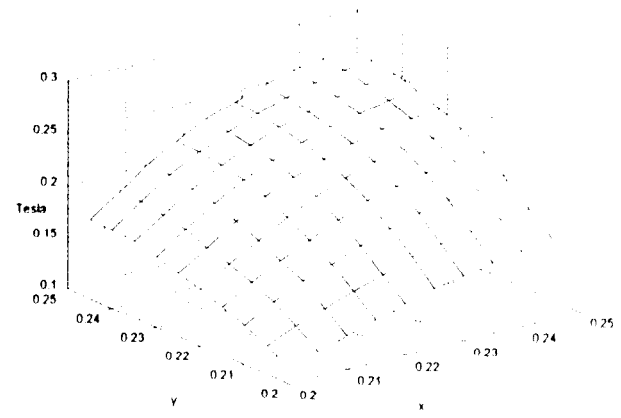


Fig. 2. Arrangement of multiple overlapping spheres to obtain the local field values on an arbitraty contour across the center points of the spheres inside a three dimensional FEM domain.



a)



b)

Fig. 3. Flux density distribution B_z on the front surface of 1 (Fig. 4) a) computed by the classical direct evaluation of the potential and b) using the new proposed post process method.

Using this local field approach (13) by arranging multiple overlapping spheres on an arbitrary surface (Fig. 2), it is possible to obtain the required local field quantities on this contour with the same accuracy when compared to the beforehand FEM computed potential values. The tetrahedron shown in Fig. 2 indicates the three-dimensional mesh of the FEM domain.

From Fig. 3 the difference between the direct evaluation of the potential and the new post process operator is shown. Here B_z is computed for a test example (Fig. 4) on the front

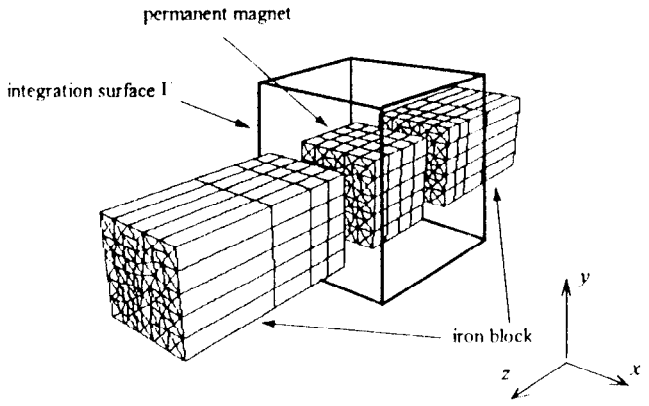


Fig. 4. Three-dimensional FEM model of the test example.

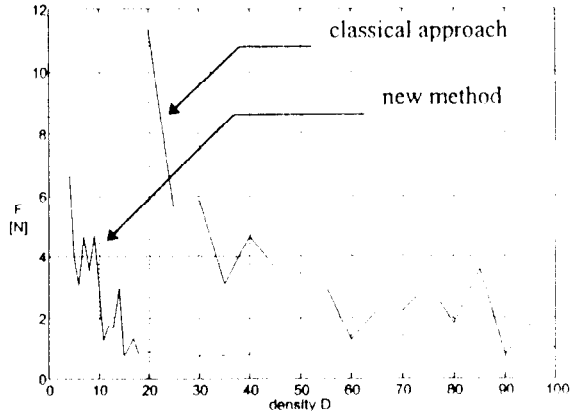


Fig. 6. Overall force pulling the permanent magnet versus density D

surface of Γ facing the permanent magnet cube. It is obvious that linear shape functions approximating the scalar potential are resulting in a piecewise constant flux density distribution (Fig. 3a). Computed forces using this type of solution are not reliable. The local values of B_z plotted in Fig. 3b show the expected continuous distribution computed using the new post processing method.

III. Three-Dimensional Test Example

To demonstrate the suitability and accuracy of the proposed method to compute local field values and forces an example including non-linear soft magnetic and permanent magnet -material is chosen. The test arrangement is symmetrical to its force axis so a zero force is the theoretically correct solution. The configuration consists of a cubic permanent magnet in the central position between two ferro magnetic blocks. Fig. 4 shows the test arrangement including the integration surface Γ to compute the overall force pulling the permanent magnet cube towards the iron blocks. The magnet is magnetised in the z-direction. The choice of the sphere parameters, the number of boundary

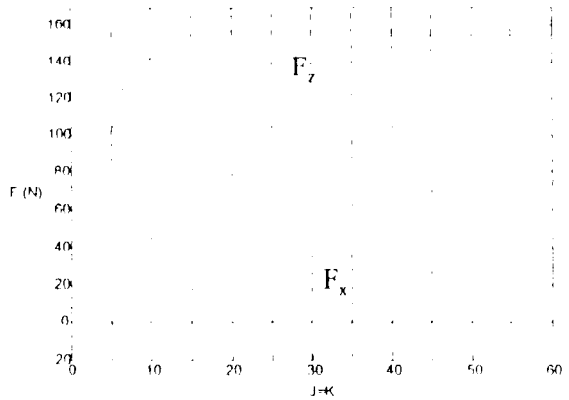


Fig. 5. Computed force pulling the permanent magnet in z-direction versus the number of boundary potentials

potential values at the surface of the arbitrary sphere J and K, are obtained by the test calculations as well. Forces are computed using the Maxwell stress tensor.

In Fig. 5 the force F_z pulling the permanent magnet to one of the iron blocks is plotted. The force is computed on the front surface of the Γ only (Fig. 5), so the resulting force in this case is not zero. Applying different numbers of J, K shows that F_z converges in a stable way towards its accurate value.

Beside the question of fixing suitable numbers for J and K the second one now arises: how many spheres, i.e. arbitrary field points inside Γ, are necessary to get the complete information out of the potential solution to represent a precise overall field distribution for the accurate computation of forces. For this purpose a density D is defined. Each side of the cubic surface Γ is now subdivided in an array of D times D equidistant points. For the new approach the sphere diameters are set to 0.02m. Using the field values in those points, the Maxwell stress tensor is applied to obtain the overall zero force of the test example. Fig. 6 indicates that a density of D=20 is sufficient to represent the total force accurately. When compared to the force pulling on each side of the magnet cube (Fig. 5), the remaining inaccuracy of 1N is of less than 0.7 %. Using the classical direct evaluation of the potentials to compute the forces a much higher density is necessary to obtain the same information. In addition, using the new proposed post process operator, the force converges towards a stable value (Fig. 6). Due to the piecewise constant flux density distribution, the values of the total force computed by the classical approach are oscillating and do not converge towards a stable and reliable solution (Fig. 6). Studies regarding the computational efforts required to obtain the full field information out of the potential solution show that with $D^2_{classical} < D^2_{new} \cdot J \cdot K$ the computation time of the classical method is slightly less when compared to the new approach. Nevertheless, the advantage of the new post process operator to supply the user with a stable and reliable solution makes this method preferable to the classical one.

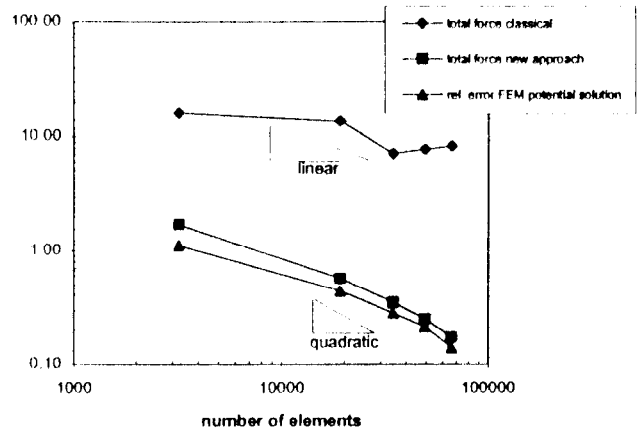


Fig. 7. Comparison of the convergence behaviour of the FEM potential solution with both, the classical post processing technique and the new proposed method.

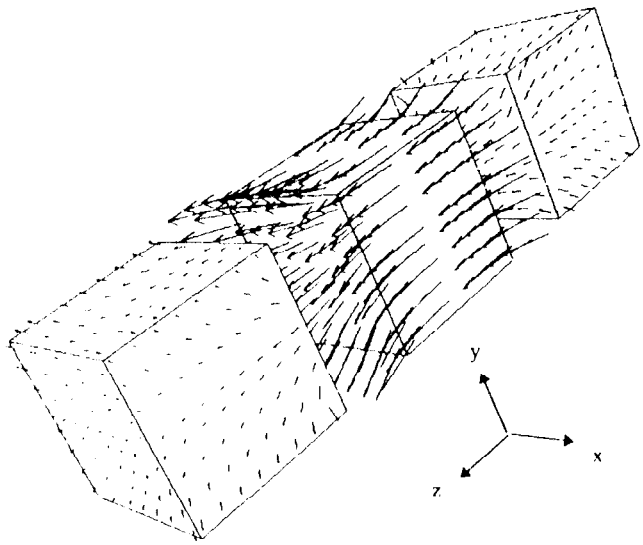


Fig. 8. Flux density plot of a three-dimensional solution for the test example.

In Fig. 7 the quadratic convergence of the FEM potential solution and the rate of convergence of the force computations using both, the classical and the new post processing approach is plotted versus the number of tetrahedron elements used. The triangles are indicating the theoretically gradient of convergence (1). The refinement of the three-dimensional discretization is performed in such a way that the elements are of the same shape in every FEM model to obtain a regularly distributed mesh for all cases. To compute the total force the Maxwell stress tensor is used integrating the partial forces calculated in points equidistantly distributed by the density D on all six sides of Γ. For the classical approach a density D=40 is chosen and in the case of the new method D is set to 7. The sphere parameters are J=K=15. The integration surface for the force computations is located in the way that no plane of Γ cuts through the nodes of the FEM mesh. If nodes are meeting the points for the force computation using the classical post processing approach this would result in a larger error due to the troublesome definition of normal and tangential field components in a node of an element. The gradient-triangles in Fig. 7 are indicating the theoretical rate of convergence for the quadratic and the linear convergence case. It can be seen and as theoretically expected that the relative error in an energy norm of the FEM potential solution converges quadratically by increasing the number of first order tetrahedron elements. Due to the analytically described potential function inside the local field volumes the resulting overall force using this approach are of the same order of convergence. Thus no loss of accuracy of the derived field quantities occurs. Looking at the convergence of the total forces computed by using the classical approach indicates a linear behaviour only. The accuracy of the obtained values are influenced by calculating the required derivatives numerically. This shows that the results obtained by the classical method are inherently inaccurate when compared to the accurateness of the potential solution.

Fig. 8 shows a three-dimensional flux density plot. The arrows indicate the direction of the flux density and their length shows the local strength of the flux density. The flux lines are representing the direction of the acting forces as well and show thus the force distribution pulling at the surface of the permanent magnet cube. The geometrical dimensions and their relation to each other, the distribution and course of the lines are an indication for the recommended three-dimensional computation of this example.

IV. Implementation

The use of the new proposed method to enhance the accuracy of computed field quantities starting from an existing potential solution demands an additional step during the post processing of the FEM analysis (Fig. 9). Having obtained a FEM potential solution the user only has to define the surface of integration Γ on which the field quantities or forces have to be calculated. The definition is performed by defining single planes in the air gap of the three-dimensional FEM model. Defining an arbitrary contour allows to compute field quantities or forces along it.

For each plane or contour the density D, the sphere parameter J, K and the radius R have to be set. The sphere parameter are problem depending and related to the geometry of the device under consideration, i.e. the air gap width. The planes or contours should be located in the middle of the air gap. A suitable value for the diameter of the single spheres is about 90-95 % of the air gap width to have as many tetrahedron finite elements inside the sphere as possible.

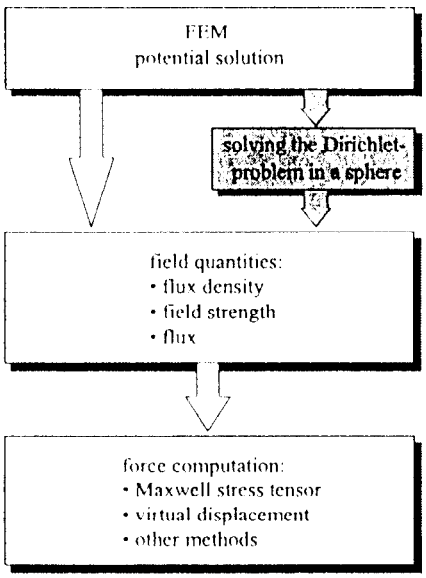


Fig. 9. Additional step during post processing to enhance the accuracy of derived field quantities.

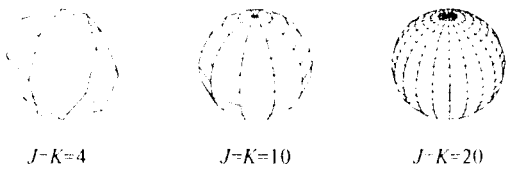


Fig. 10. Spheres with different numbers for the parameter J, K.

Including only one finite element in the sphere results in no enhancements in accuracy of the derived quantity.

The density D should be chosen in such a way that the spheres are overlapping (Fig. 2). For the distance between two points on the surface of integration it is suitable to choose the radius of the sphere as the value.

To define the number and position of boundary potential values distributed on the surface of each sphere the parameters J and K have to be chosen. To ensure uniformly distributed boundary values J is set equal to K. In accordance with the computations plotted in Fig. 6 and other test calculations a number $J=K=[10 \ldots 20]$ is sufficient to meet the ratio between computational costs and accuracy. Fig. 10 illustrates by different $J=K$ the position and number of boundary potentials to approximate the local field inside a sphere.

V. Conclusions

The local solution of the Laplace equation (2) inside the air gap of an electromagnetic device using a Fourier series approximation as potential function results in a significant increase of accuracy of the derived field quantities.

Inherent inaccurate solutions obtained by numerically derivated field values are caused by the loss of one order of rate of convergence when the derived quantities are compared to the potential solution. This problem occurs in two- and three-dimensional field problems. A solution of the 2D problem is given in [2]-[4]. Following the same basic idea the local post process method introduced here, describes a technique to enhance the accuracy of the results i.e. field values and forces, derived from a potential solution in three-dimensions. Here, a finite element solution applying standard linear tetrahedron elements is used. Solving a local Dirichlet problem inside a sphere analytically enhances the accuracy of the derived field quantities of a three-dimensional finite element model. A Fourier series is used to represent the potential function of the local field problem. Necessary parameters are introduced and suitable numbers are given. With this approach it is shown that the rate of convergence for the field solution is the same as that of the FEM potential solution itself. A three-dimensional FEM test example including permanent magnet material demonstrates the behaviour of the proposed method. Results for local field values and computed forces point out its suitability. The comparison of computed forces using the classical post processing approach and the new post process operator show the advantage concerning accuracy and stability of solutions obtained by the proposed method.

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