ENHANCED ACCURACY OF 2D FINITE ELEMENT FIELD QUANTITIES BY A LOCAL POST-PROCESS

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ABSTRACT

Energy conversion in electromagnetic energy transducers takes place in the air gap only. Numerical field computation techniques of general application range are used for the design and optimisation of those electromagnetic devices. To predict the operational behaviour of such devices, particular attention has to be paid to the computation of the air gap values of the flux densities and the magnetic field strength. Ongoing publications and research activities on force computations in electromagnetic devices using different approaches indicate the importance of this field. This paper is focused on the practical application of the static electromagnetic field solution of the LAPLACE equation in a local post-process to increase the accuracy of an existing solution obtained by the standard two-dimensional finite element method using triangular types of elements. Advantages and drawbacks are discussed.

INTRODUCTION

Using a two dimensional finite element approach, the vector potential $\boldsymbol{A}$ is computed and the interesting magnetic field quantities are derived from this solution. The numerical evaluation of derivatives can be troublesome due to round off errors during the calculation. The flux density $\boldsymbol{B}$ is determined by the partial derivatives of $\boldsymbol{A}$.

$$\mathbf{B} = curl\mathbf{A} \tag{1}$$

Therefore, the resulting flux density looses one order in accuracy when compared to the order of accuracy of the vector potential. Using standard linear shape functions to approximate the continuous vector potential over a triangular finite element results in an only piece wise constant magnetic flux density. As a consequence, the calculated forces from those derived quantities are less accurate when compared to the vector potential results. This causes numerical errors due to numerical integration by using the MAXWELL stress tensor to evaluate globally generated forces.

For a more accurate force calculation the aim is to improve the results of an existing field solution in a local post-process. The idea is to solve the LAPLACE equation

$$\nabla^2 u = 0 \tag{2}$$

in source free and homogenous areas, e.g. in the air gap of an electromagnetic device starting with an existing field solution. As a method, the local potential is approximated by a FOURIER series using the existing solution as boundary values of the local problem formulation.

FORCE COMPUTATION

Different methods to compute forces are in common use (2, 3). The MAXWELL stress tensor and the principles of the virtual work are the most frequently used.

Maxwell Stress Tensor

To compute the forces according to the MAXWELL stress tensor, the local values of the normal component of the flux density B_n and the tangential component of the magnetic field strength H_t are necessary and therefore, the derivatives of the vector potential have to be evaluated. The total force is obtained by a sum of local forces along an arbitrary closed surface Γ. Applying $\mathbf{t}$ as the tangential unit vector respectively $\mathbf{n}$ as the normal unit vector and μ_0 the permeability of free space, in two dimensions the MAXWELL stress tensor is given by

$$F_M = \oint_\Gamma \left\{ (B_n H_t)\mathbf{t} + \frac{1}{2}\left(\frac{1}{\mu_0} B_n^2 - \mu_0 H_t^2\right)\mathbf{n} \right\} d\Gamma \tag{3}$$

To obtain reliable force predictions calculated by eq. (3) the need for the highly accurate computed electromagnetic field quantities is obvious.

Virtual Work Principle

For the principle of the virtual displacement two field evaluations are required to compute the forces via the change of co-energy stored in the finite element model. The displacement is performed assuming constant flux linkage Ψ. To hold this condition only small displacements in the direction ρ are allowed. Using this approximation the total force is expressed by

$$F_{vd} = -\left.\frac{\partial W}{\partial \rho}\right|_{\Psi=\text{const}} \tag{4}$$

In this paper the MAXWELL stress tensor is used as only one field solution is necessary.

LOCAL SOLUTION OF THE LAPLACE EQUATION

The LAPLACE eq. (2) written in polar coordinates (r,Φ) is:

$$\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)+\frac{1}{r}\frac{\partial^2 u}{\partial \Phi^2}=0$$

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \Phi^2}=0 \tag{5}$$

The idea now is to find a function which represents formally an infinite series in which every term is a solution of the differential equation and the boundary values are satisfied. Assuming linearity and uniformity of the LAPLACE equation and thus applying to eq. (5) a FOURIER approach, given in Meis et al (4), leads to the harmonic function:

$$u(r,\Phi)=\frac{\alpha_0}{2}+\sum_{n=1}^{\infty} r^n\left\{\alpha_n\cos(n\Phi)+\beta_n\sin(n\Phi)\right\} \tag{6}$$

with its coefficients α_n and β_n .

$$\alpha_n=\frac{1}{\pi R^n}\int_0^{2\pi} u(R,\Phi)\cdot\cos(n\Phi)\,d\Phi$$

$$\beta_n=\frac{1}{\pi R^n}\int_0^{2\pi} u(R,\Phi)\cdot\sin(n\Phi)\,d\Phi \tag{7}$$

The procedure to solve eq. (6) describes the solution of a DIRICHLET problem on a circle with given boundary values on its circumference. Equation (7) represents a FOURIER series and the coefficients α_n and β_n can be calculated by using the known potentials $u=u(R,\Phi)$ on the circumference of a circle with radius R .

Now a finite number of N equiangularly ordered points is applied onto the circumference of the circle.

$$u_i(R,\Phi_i)=u(R,i\cdot\tfrac{2\pi}{N}) \quad , i=1(1)N \tag{8}$$

With N boundary potential values u_i known on the circumference and according to the properties of harmonic functions the first term in eq. (6) can be written as:

$$u|_{r=0}=\frac{\alpha_0}{2}=\frac{1}{N}\sum_{i=1}^{N} u_i \tag{9}$$

The Fourier coefficients are rewritten as follows:

$$\alpha_n=\frac{2}{N\cdot R^n}\sum_{i=1}^{N} u_i\cos(n\Phi_i)$$

$$\beta_n=\frac{2}{N\cdot R^n}\sum_{i=1}^{N} u_i\sin(n\Phi_i) \tag{10}$$

With the Fourier series eq. (6) and their coefficients eqs. (9, 10) the potential in the centre of a circle can be computed by only knowing the boundary potential values on the circumference of the circle.

Using this approach inside a finite element solution, the value of the potential of a field point is now depending on the solution in several finite elements. Thus, local numerical errors in single elements will have a relatively small influence on the solution in the considered field point. Applying eq. (6) derivatives at the centre of the circle can be calculated.

The idea now is to transform the described process, of solving a DIRICHLET problem on a circular surface, to determine the vector potential in a point P_1 of a discretised finite element domain. Figure 1 illustrates this idea. R_1 is the radius of the considered circular surface and the dots on the circumference indicate the points of known vector potential values computed by the existing finite element solution.

To obtain the potential on a given contour inside a finite element domain, multiple circles have to be evaluated. Overlapping circles guarantee a continuous solution in the considered region after this post-process.

The numerical shape of the eqs. (6, 9, 10) enables an easy implementation of the procedure in a finite element program package. Advantageous is the shape of the contour

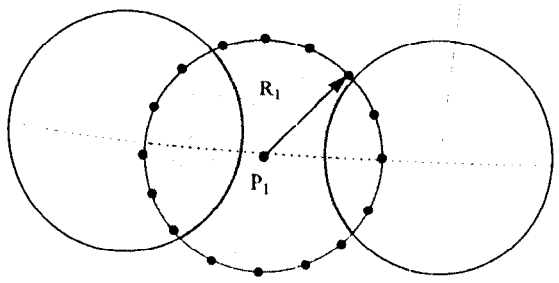

Figure 1: Multiple circles determine the potentials on a contour

eq. (6). The derivatives in the centre of the circle are represented by the FOURIER coefficients. Thus, no additional computational efforts are necessary to compute

$$\left.\frac{\partial u}{\partial x}\right|_{r=0} = \alpha_1 = \frac{2}{N \cdot R}\sum_{i=1}^{N} u_i \cos\Phi_i$$

$$\left.\frac{\partial u}{\partial y}\right|_{r=0} = \beta_1 = \frac{2}{N \cdot R}\sum_{i=1}^{N} u_i \sin\Phi_i \qquad (11)$$

APPLICATION OF THE METHOD

Using the standard finite element method in two dimensions, the field area is discretised into non-overlapping triangular elements with homogeneous material properties. Applying simple linear shape functions to the single elements to approximate the vector potential, a continuous potential distribution is obtained. With eq. (1) the flux density distribution is piece wise constant. Compared to the vector potential A, the approximation of the flux density distribution is of one order lower. To surmount this disadvantage, an additional step in the post-process is applied. Figure 2 illustrates this additional step during the post-process. Using the local solution of the LAPLACE equation on a circle, applying the known potentials as boundary values, increases the accuracy of the already computed field values in this region.

The local solution of the LAPLACE equation inside an air gap of an electromagnetic device using a FOURIER series approximation for the vector potential results in a significant increase of accuracy of the derived field quantities. Investigations concerning the convergence rate of the solution can be found in Kasper and Franz (1). To compare the results obtained by the local field evaluation to the conventional obtained field quantities of first order elements fig. 3 shows the computed vector potential **A** and the derived magnetic flux density in direction x of the Cartesian coordinate system with and without using the LAPLACE approach. For the application of the local Dirichlet problem 24 potential boundary values on the circumference of the circle are used.

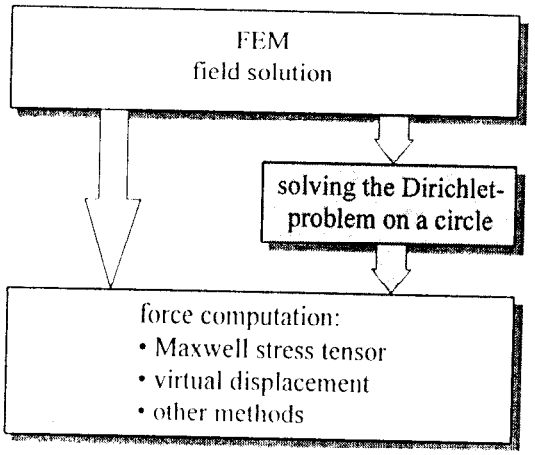

Figure 2: Additional step during post-processing to enhance the accuracy of existing field

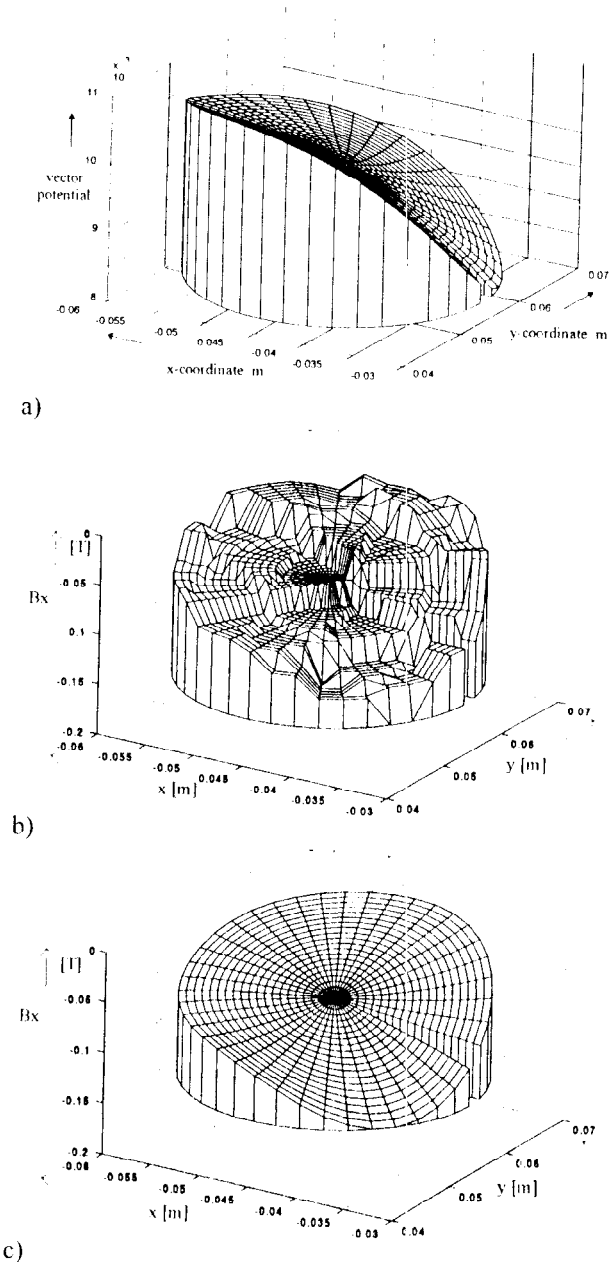

Figure 3: a) Computed vector potential inside the circular FEM domain, b) the derived magnetic flux density B_x and c) the resulting flux density B_x derived

Figure 3a shows the local vector potential of the existing FEM solution inside the circular domain. The resulting magnetic flux density using eq. (1) is plotted in fig. 3b. For the finite element discretisation linear elements of first order were used. It is obvious that computed forces using this type of solution are not reliable. In fig. 3c the local values of the flux density obtained by using the solution of the described DIRICHLET problem are plotted. Using the analytical derivatives of eq. (6), it can be seen that the order of this solution is of the same as the order of the solution of the vector potential (fig. 3a).

RESULTS

The method introduced is applied to a magneto static field problem. Computations are performed on a test example consisting of a diametrical magnetised circular permanent magnet and a ferromagnetic back iron yoke (fig 4). With the very dense mesh plotted in fig 4a an accurate flux density solution can be expected using eq. (1). Due to symmetry the global force is zero. In fig. 4b the computed flux density distribution is plotted. Due to the magnetisation direction of the permanent magnet, an angular sinusoidal normal component of the flux density distribution inside the air gap is expected. Using the local solution computations are performed on a median circular contour in the air gap. The radius of the single local circles is set to the value of half of the air gap length. Thus, the local circles fill the whole air gap region.

Due to the high discretisation of the finite element domain (fig. 4a) a good agreement between the results obtained with the evaluation according eqs. (1) and (6) is found. The computed field quantities plotted in fig. 5a are the normal- and the tangential-component (fig. 5b) of the flux density. The flux density obtained by the conventional method, direct evaluation of eq. (1), scatter around the values obtained by the application of the FOURIER approach as illustrated in fig. 5. Kasper and Franz (1) report of a doubling of the correct and significant digits in the results when compared to the usual FEM approximation. This is caused by a higher rate of convergence.

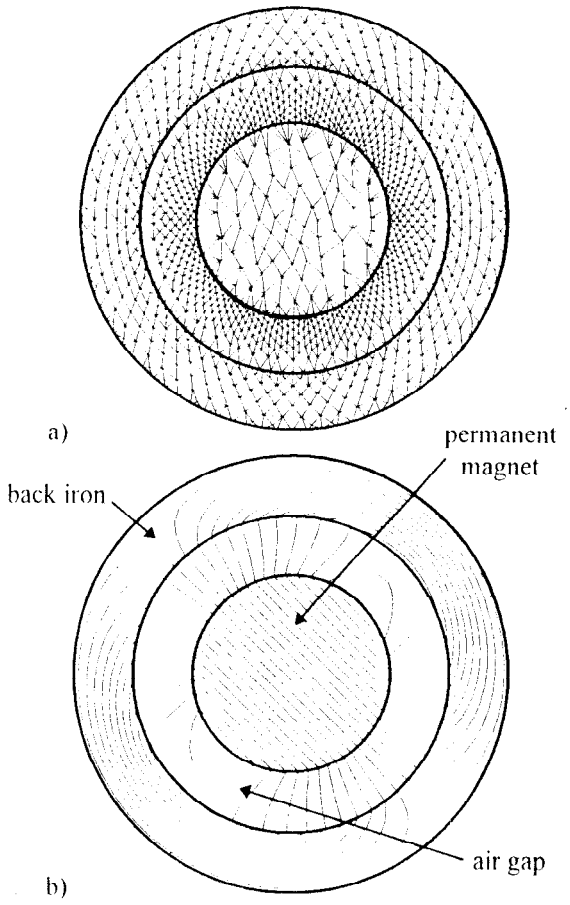

Figure 4: a) FEM mesh and b) flux plot of the test example.

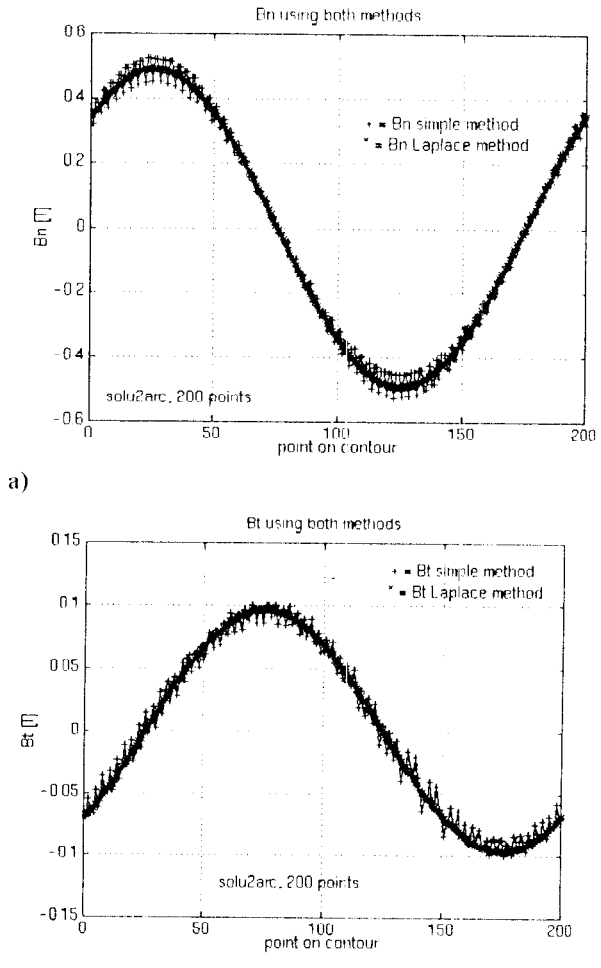

Figure 5: a) Normal- and b) tangential-component of the flux density distribution in the air gap computed using eq. (1) and the local post-process eq. (6).

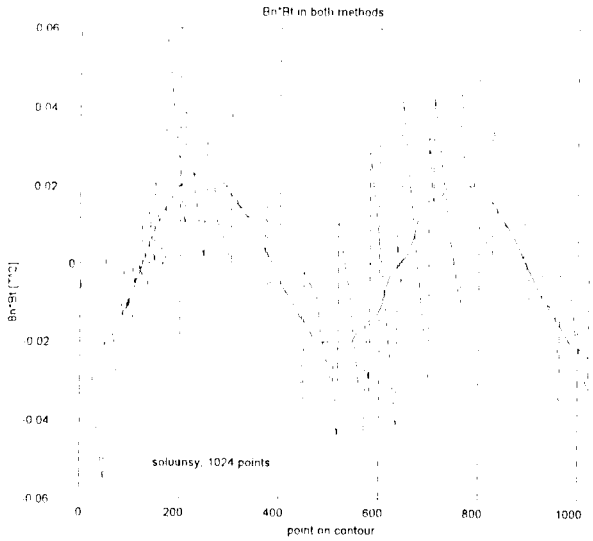

Figure 6: Comparison of the product $B_n \cdot B_t$ vs. angular position, computed at the median contour in the air gap using both methods.

Computing the force by integrating the Maxwell stress tensor along the median contour in the air gap, the product of normal- and tangential-component of the flux density is necessary. Applying both approaches, the product $B_n \cdot B_t$ is calculated.

A comparison of the results obtained by both methods is shown in fig. 6. The values obtained by the conventional post-process show large deviations around the values computed by the local post-process approach. The total numerical error of a finite element approximation consists of the discretisation error and additional rounding errors (1). Using the dense mesh from fig. 4a the discretisation error is relatively small. In spite of the relatively good approximation of the single B_n, B_t components of the magnetic flux density, caused by the dense finite element mesh, the product of both quantities shows high inaccuracies. Thus, the numerical error using the conventional post-process approach is higher when compared to the error using the local post-process routine. It can be found that by integrating the values along the median contour in the air gap the theoretical expected zero force is very good approximated by using the local post-processing technique.

To avoid numerical errors by applying the local post-process approach, the radius of the circles must be as large as possible to include as many finite elements as possible. In several test calculations, performed with different finite element meshes, it is found that the diameter of the circles must be at least larger than the mesh size. This ensures an increased accuracy of the results when compared to the conventional post-processing. As a consequence, using a less dense finite element mesh for the local post-process approach, results in field quantities of the same accuracy as obtained by the conventional post-process applied to a very high discretised finite element domain.

CONCLUSIONS

The local post-process approach introduced describes a technique to enhance the accuracy of the results of an already existing finite element solution. Using this approach reliable force calculations are possible. The known values of the vector potential are boundary values for the local field problem. A FOURIER series is used to approximate the vector potential in the local finite element domain. The necessarily additional step during the post-process is easy to implement in the finite element software. Compared to the conventional post-processing an acceptable compromise between computational costs and accuracy is found.

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