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Electrodynamic Finite Element Model Coupled to a Magnetic Equivalent Circuit*

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Abstract. An electrodynamic field is coupled to a magnetic equivalent circuit. The electrodynamic problem is formulated by the electric vector potential and discretised by finite elements. The magnetic lumped parameter model is described in terms of unknown fluxes and magnetomotive forces. The coupled system matrix has a mixed and hybrid nature. In this presentation, the method is applied to simulate eddy current distributions in laminated material and losses in a dielectric heater.

Résumé. Un champ électrodynamique est couplé à un circuit magnétique équivalent. Pour le problème électrodynamique, on utilise une formulation en potentiel vecteur électrique, dicrétisée par éléments finis. Les inconnues du circuit équivalent sont les flux et les forces magnétomotrices. La matrice du système couplé est mixte et hybride. La méthode est appliquée à la simulation de la distribution des courants induits dans un matériau laminé, et au calcul des pertes d'un système de chauffage diélectrique.

Running title. Electrodynamic FEM coupled to MEC

PACS. 02.60.Cb Numerical simulation; solution of equations - 02.70.Dh Finite-element and Galerkin methods - 84.50.+d Electric motors

1 Introduction

Maxwell's laws indicate a strong liaison between the electric and the magnetic field. In almost every model of an electrotechnical device, this coupling has to be considered. As a consequence, electromagnetic simulation has a *mixed* nature: both electric and magnetic quantities appear as unknowns and have to be computed simultaneously. To turn from the continuous model over to the discrete one, several discretisation techniques, such as equivalent circuits, finite elements and boundary elements are applicable. As electric and magnetic phenomena are linearly related to each other, the discrete coupling of both fields is easily realized in one system matrix. If both fields are discretised by different methods, a *hybrid* model is achieved. Common examples are the simulations of electrical motors. The quasi-static electric fields are characterised by linear material characteristics and current distributions following clearly determined paths through the conductive parts in the model. The magnetic fields, however, suffer from arbitrary flux paths and non-linear material properties. As a consequence, this type of technical devices is efficiently and accurately modelled by a magnetic finite element model coupled to an electric lumped parameter description. The technical importance of this kind of hybrid coupling schemes is reflected by the large efforts found in literature to optimise field-circuit coupling simulation techniques [1].

There are also devices, e.g. laminated materials, induction furnaces and dielectric heaters for which this assumptions are not true. The electrodynamic field requires an accurate description whereas the magnetic field can be represented by a magnetic equivalent circuit. This paper considers field-circuit coupled electrodynamic-magnetic models.

2 Electrodynamic finite element model

The continuity of the current density $\mathbf{J}$ and Ampère's law for the magnetic field strength $\mathbf{H}$ are applied by defining the electric vector potential $\mathbf{T}$ by $\mathbf{J} = \nabla \times \mathbf{T}$ and the magnetic scalar potential φ by $\mathbf{H} = \mathbf{T} - \nabla\varphi$. The combination of the constitutive relations with Faraday-Lenz's law yields the governing differential equation

$$\nabla \times (\rho \nabla \times \mathbf{T}) + j\omega\mu\mathbf{T} = j\omega\mu\nabla\varphi . \quad (1)$$

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Table 1: Coefficients of the discrete system of equations.

		2D	axisymmetric
Diffusive term	k_{ij}	$\int_\Omega \rho \nabla N_i \cdot \nabla N_j dxdy$	$\int_\Omega \frac{\rho}{2\pi r} \nabla N_i \cdot \nabla N_j drdz$
Convective term	l_{ij}	$\int_\Omega \mu N_i N_j dxdy$	$\int_\Omega \frac{\mu}{2\pi r} N_i N_j drdz$
Load	f_i	$\int_\Omega \mu \nabla \varphi N_i dxdy$	$\int_\Omega \mu \nabla \varphi N_i drdz$
Unknown	ψ_j	T_{zj}	I_{potj}
Magnetic conductivity	G_m	$\int_\Omega \frac{\mu}{\ell_z} dxdy$	$\int_\Omega \frac{\mu}{2\pi r} drdz$
Coupling to MEC	q_{mj}	$\int_\Omega \frac{\mu}{\ell_z} N_j dxdy$	$\int_\Omega \frac{\mu}{2\pi r} N_j drdz$

ω is the pulsation, ρ the resistivity and μ the permeability of the applied materials.

The geometry and the excitation of the devices considered here, permit a 2D or axisymmetric discretisation. In that case, the current is in the plane (*x-y*-plane or *r-z*-plane respectively) whereas the flux and the electric vector potential are perpendicular to the plane. In 2D, the vector potential has only a *z*-component T_z:

$$-\frac{\partial}{\partial x}\left(\rho \frac{\partial T_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\rho \frac{\partial T_z}{\partial y}\right) + j\omega\mu T_z = j\omega\mu\nabla\varphi \,. \quad (2)$$

The current through a surface formed by the *z*-extrusion of a curve $(x_1, y_1) \to (x_2, y_2)$ in the *xy*-plane is

$$I_{2D} = \ell_z \left(T_z(x_2, y_2) - T_z(x_1, y_1)\right). \quad (3)$$

In the axisymmetric case, the vector potential consists of the tangential component T_θ:

$$-\frac{\partial}{\partial r}\left(\frac{\rho}{r} \frac{\partial (r T_\theta)}{\partial r}\right) - \frac{\partial}{\partial z}\left(\rho \frac{\partial T_\theta}{\partial z}\right) + j\omega\mu T_\theta = j\omega\mu\nabla\varphi \,. \quad (4)$$

The current through a surface formed by rotating $(r_1, z_1) \to (r_2, z_2)$ along the axis is

$$I_{ax} = 2\pi r_2 T_z(r_2, z_2) - 2\pi r_1 T_z(r_1, z_1). \quad (5)$$

For axisymmetric simulation, it is convenient to use the current between a point (r, z) and the axis

$$I_{pot} = 2\pi r T_z(r, z) \quad (6)$$

as the potential in the partial differential equation [2]:

$$-\frac{\partial}{\partial r}\left(\frac{\rho}{2\pi r} \frac{\partial I_{pot}}{\partial r}\right) - \frac{\partial}{\partial z}\left(\frac{\rho}{2\pi r} \frac{\partial I_{pot}}{\partial z}\right) + j\omega \frac{\mu}{2\pi r} I_{pot} = j\omega\mu\nabla\varphi \,. \quad (7)$$

The domain Ω of the electrodynamic model is discretised by linear triangular elements N_i. The same functions are applied as weighting functions in the Galerkin finite element approach. The discrete system of equations is

$$\left[k_{ij} + j\omega l_{ij}\right]\left[\psi_j\right] = \left[j\omega f_i\right], \quad (8)$$

with the coefficients defined in Table 1.

The system of equations is complex, sparse and symmetric with typical sizes of 10000 up to a few 100000 unknowns. Modern Krylov subspace solvers, such as Conjugate Orthogonal Conjugate Gradients (COCG) and Quasi Minimal Residual (QMR) combined with an appropriate preconditioner are very effective to solve these systems [3].

3 Global magnetic quantities

The magnetomotive force (MMF) V_m along a flux path $\ell_1 - \ell_2$ is defined by

$$V_m = \int_{\ell_1}^{\ell_2} (-\nabla\varphi) d\ell \,. \quad (9)$$

The load of the electrodynamic finite element model is related to the MMF by

$$-\nabla\varphi = \frac{V_m}{\ell_z} \,. \quad (10)$$

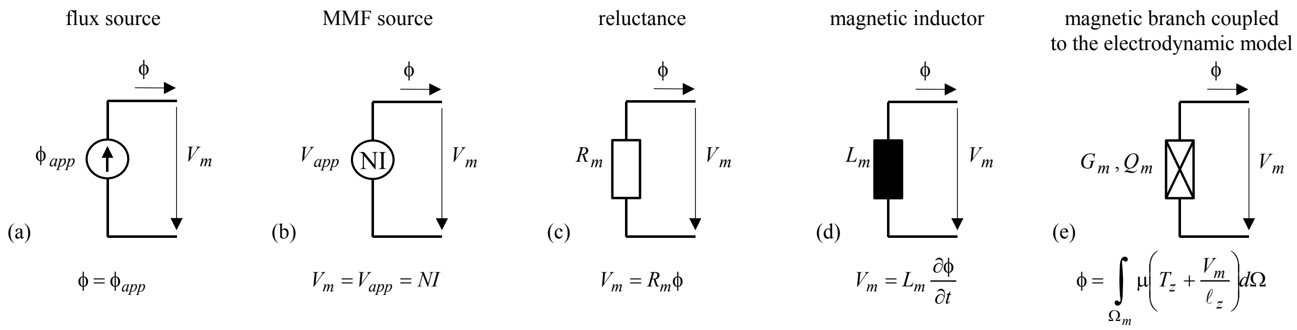

Fig. 1: Magnetic circuit elements.

for 2D models and

$$-\nabla\varphi = \frac{V_m}{2\pi r}\,. \tag{11}$$

for axisymmetric models. An axisymmetric model that is symmetric in the tangential direction for $0 \le \theta \le \alpha$ with α an opening angle different from 2π has a limited technical relevance and is not considered here. The MMFs are introduced as extra unknowns in the model.

The magnetic flux through a part Ω_m of the computational domain is

$$\phi_{2D} = \int_{\Omega_m} \left(\mu \frac{V_m}{\ell_z} + \mu T_z \right) d\Omega\,; \tag{12}$$

$$\phi_{ax} = \int_{\Omega_m} \left(\mu \frac{V_m}{2\pi r} + \mu \frac{I_{pot}}{2\pi r} \right) d\Omega\,. \tag{13}$$

These integral relations bring the magnetic flux into relation to the MMF and the electric vector potential distribution. The discrete form of (12) and (13) is

$$\phi = G_m V_m + \lfloor q_{mj} \rfloor \lfloor \psi_j \rfloor, \tag{14}$$

with the coefficients defined in Table 1. G_m denotes the magnetic conductivity whereas $\lfloor q_{mj} \rfloor \lfloor \psi_j \rfloor$ can be seen as a flux source controlled by the electric current distribution.

When comparing the magnetic branches defined here, to the solid conductor model in magnetic finite element models, both correspondences and differences are observed. In both cases, a driving force (MMF or electric voltage) is applied as an extra unknown. A flow (magnetic flux or electric current) is related to the force and the (electric or magnetic) vector potentials by a discretised integral expression. A difference lays in the fact that solid conductors experience induced currents due to the time-derivative in Faraday-Lenz' law whereas magnetic conductors experience excited fluxes (Ampère's law does not contain a time-derivative). The time-derivative appears in the load of the electrodynamic model.

4 Coupling to a magnetic equivalent circuit

The magnetic equivalent circuit (MEC) consists of flux sources, MMF sources, reluctances, magnetic inductors and the magnetic branches embedded in the electrodynamic model (Fig. 1). A current driven inductor is represented by a MMF source $V_{app} = NI$ with N the number of turns and I the applied current (Fig. 1b). A voltage driven inductor is modelled as a flux source $\phi_{app} = -V/j\omega$ with V the applied voltage (Fig. 1a). The reluctances of external magnetic paths are represented by passive reluctance elements (Fig. 1c). Shading rings as constructed in split-pole one-phase induction machines and actuators can be modelled by magnetic inductances (Fig. 1d). They introduce the phase-lag between the applied MMF source and the induced magnetic flux. The magnetic branches that are part of the electrodynamic model (Fig. 1e) are described by unknown MMFs and the integral relations (12) and (13). An equivalence can be observed between the elements of the MEC and the elements forming the electrical circuit coupled to a magnetic finite element model. Only for the electrical capacitors and the stranded conductor model, the correspondences to some magnetic elements is not obvious.

The circuit model is arranged into a system of equations by choosing an appropriate set of unknowns and selecting their corresponding equations. The coupling to the electrodynamic finite element model follows automatically when the magnetically coupled branches are embedded in the MEC and treated as such. If no MMF sources and magnetic inductors occur in the MEC, a common modified nodal analysis of the circuit part is sufficient to obtain a sparse and symmetric coupled system. Here, a more general topological method is applied. The treatment is similar to the approach presented in [4] and ends up with a mixed description in terms of both unknown MMFs and fluxes aside the electric vector potentials. A tree is traced through the MEC. The tree consists of a set of branches connecting all nodes of the circuit without forming loops. The branches selected to participate in the tree are in order of priority: MMF sources, branches coupled to the electrodynamic field, reluctances, magnetic inductors and flux sources. The tree branches are modelled by MMFs. The MMFs of the MMF sources appear as loads in the righthandside. For the other branches, unknown MMFs are introduced. The remaining branches are links and form the cotree. To these branches, except to the flux sources, unknown fluxes are assigned.

To each tree branch corresponds a fundamental cutset, i.e. the combination of the tree branch with a unique set of links which removal divides the circuit in two parts. A general form of Kirchhoff's current law is applicable to the fundamental cutset and relates the flux through the tree branch to the known and unknown fluxes through the links:

$$[\phi_t] + \mathbf{D}[\phi_l] = 0 . \qquad (15)$$

with $\mathbf{D}$ the fundamental cutset matrix. The branch relations for the tree branches relate the fluxes ϕ_t to the unknown MMFs v_t :

$$[\phi_t] = \mathbf{G}_m[v_t] + \lfloor q_{mj} \rfloor [\psi_j] . \qquad (16)$$

$\mathbf{G}_m$ represents the magnetic admittances of the reluctances and coupled branches present in the tree. In the case of a coupled electrodynamic-magnetic branch, this operation introduces the coupling terms in the system matrix.

For the fundamental loops, i.e. the loops formed by a link and a unique set of tree branches, the Kirchhoff voltage law is expressed by

$$[v_l] + \mathbf{B}[v_t] = 0 . \qquad (17)$$

with $\mathbf{B}$ the fundamental loop matrix. The voltages v_l are related to the fluxes ϕ_l by the branch relations of the links:

$$[v_l] = \mathbf{Z}_m[\phi_l] . \qquad (18)$$

$\mathbf{Z}_m$ represents the magnetic impedances of the reluctances and inductances that are part of the cotree.

The coupled system matrix is

$$\left[\begin{array}{c:cc} k_{ij} + j\omega l_{ij} & j\omega q_{it} & 0 \\ \hdashline j\omega q_{sj} & \xi \mathbf{G}_m & \xi \mathbf{D} \\ 0 & -\xi \mathbf{B} & -\xi \mathbf{Z}_m \end{array}\right] \left[\begin{array}{c} \psi_j \\ \hdashline v_t \\ \phi_l \end{array}\right] = \left[\begin{array}{c} 0 \\ \hdashline \xi \phi_{app} \\ -\xi V_{app} \end{array}\right] . \qquad (19)$$

The fundamental property of circuit theory $\mathbf{D} = -\mathbf{B}^{\mathrm{T}}$ and the application of the scaling factor $\xi = \xi_{2D} = j\omega / \ell_z$ or $\xi = \xi_{ax} = j\omega$ symmetrises the external circuit equations with respect to the finite element equations. Also, the coupling mechanism preserves the sparsity of the original finite element system. COCG and QMR are still the methods of choice. The bordered form of the matrix may, however, bring up a worse convergence. Block-preconditioning or strong coupled multigrid preconditioning may be required to achieve acceptable solution times [5].

The topological method, introduced here, failes in the case of circularly connected electrically coupled branches or star-connected magnetic inductors. In that case, magnetic inductors occur in the tree and/or coupled branches participate to the cotree. A clear assignment of unknowns is troublesome. The partial cutset and loop eliminations curing this problem are described in [4] and are beyond the scope of this paper.

5 Eddy currents in laminations

The first example consists of several iron laminates with coating material on both surfaces (Fig. 2). The coating material is less conductive and less permeable than the iron, preventing excessive eddy current losses but at the expense of a higher reluctance of the global magnetic flux path [6]. Semi-analytical simulations consider the losses in a single laminate and neglect the conductivity and permeability of the coating material. The model presented here deals with coating material with a finite resistivity. As a consequence, the closing path of the current may cross the coating layers. The eddy current losses are completely different from the simplified analytical model. As an external condition, the total magnetic flux through the model has to equal the applied flux. The MEC represents the parallel connection of all domains in the electrodynamic model, excited by a flux source (Fig. 2).

6 Coupling to a magnetic equivalent circuit

The second example is a dielectric heating device (Fig. 3). A cylindrical dielectricum is placed between two circular electrodes. Both dielectric and conductive heating effects are considered [7]. The geometry and the excitation are modelled by an axisymmetric model. As a consequence, the MEC consists of the short-circuit connection of all magnetic paths. Here, the excitation has an electric nature and is applied as a difference in electric vector potential. The combination of conductive and dielectric effects involves a complex valued resistivity in (1). If the geometrical dimensions exceed the wave length, a wave phenomenon is observed (Fig. 3).

7 Conclusions

An electrodynamic finite element model is combined with a magnetic equivalent circuit in one sparse and symmetric system matrix. The coupling scheme is applied to simulate the eddy current losses in laminated material and the total losses in a dielectric heating device.

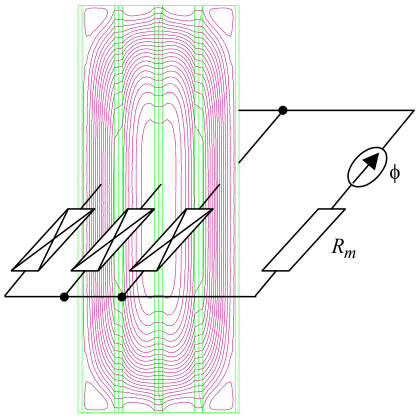

Fig. 2: 2D electrodynamic model of a laminated material combined with a magnetic equivalent circuit.

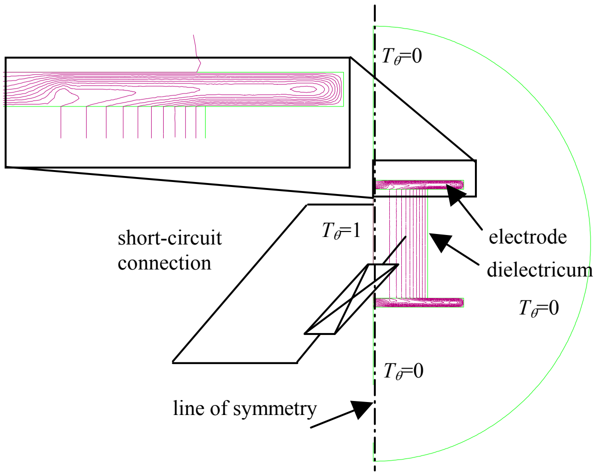

Fig. 3: 2D electrodynamic model of a laminated material combined with a magnetic equivalent circuit.

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