Algebraic Multigrid for Complex Symmetric Systems

D. Lahaye, H. De Gersem, S. Vandewalle, and K. Hameyer

***Abstract*—The two dimensional quasistatic time-harmonic Maxwell formulations yield complex Helmholtz equations. Multigrid techniques are known to be efficient for solving the discretization of real valued diffusion equations. In this paper these multigrid techniques are extended to handle the complex equation. The implementation of geometric multigrid techniques can be cumbersome for practical engineering problems. Algebraic multigrid (AMG) techniques on the other hand automatically construct a hierarchy of coarser discretizations without user intervention given the matrix on the finest level. In the linear calculation of an induction motor the use of AMG as precondioner for a Krylov subspace solver resulted in a six-fold reduction of the CPU time compared to an optimized incomplete LU factorization and in a twenty-fold reduction compared to symmetric successive overrelaxation.**

***Index Terms*—Eddy current, iterative methods.**

I. Introduction

SINUSOIDALLY excited eddy current problems arise, e.g., in the design and optimization of induction furnaces, transformers and alternating current machines, such as induction motors. They are commonly treated as single frequency time-harmonic magnetic field problems [1]. Their finite element discretizations result in sparse, complex symmetric systems of equations. Solving such systems often absorbs more than 90% of the overall CPU-time of the numerical simulation [2]. Hence, an investigation of iterative methods specific for this kind of matrices is appropriate.

II. Time-Harmonic Magnetic Formulations

Assuming that the electromagnetic quantities are varying at low frequency and sinusoidally in time, and that the geometry is two dimensional, the Maxwell equations can be simplified to a scalar Helmholtz equation with complex shift [1]:

$$-\frac{\partial}{\partial x}\left(\nu\frac{\partial \hat{A}_z}{\partial x}\right)-\frac{\partial}{\partial y}\left(\nu\frac{\partial \hat{A}_z}{\partial y}\right)+\boldsymbol{j}\omega\sigma\hat{A}_z=\hat{J}_s. \quad (1)$$

In this equation $\hat{A}_z$, ν, $\boldsymbol{j}$, ω, σ, and $\hat{J}_s$ are the phasor of the z-component of the magnetic vector potential, the reluctivity, the complex unit, the angular frequency, the conductivity and the phasor of the applied source current density.

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We solve equation (1) numerically using linear triangular finite elements. The exact solution $\hat{A}_z$ is approximated by a linear combination of real nodal form functions N_i

$$\hat{A}_h=\sum_i x_i N_i, \quad (2)$$

where the complex coefficients x_i in the expansion have to be determined. These coefficients are calculated by solving the system of algebraic equations

$$\boldsymbol{A}\boldsymbol{x}=\boldsymbol{b}, \quad (3)$$

with

$$\boldsymbol{A}=\boldsymbol{A}_R+\boldsymbol{j}\boldsymbol{A}_I, \quad (4)$$

where the elements of the real and imaginary part $\boldsymbol{A}_R$ and $\boldsymbol{A}_I$ of the matrix $\boldsymbol{A}$ are given by

$$\boldsymbol{A}_{R,kl}=\int_\Omega \nu\nabla N_k\cdot\nabla N_l\,d\Omega \quad (5)$$

and

$$\boldsymbol{A}_{I,kl}=\int_\Omega \omega\sigma N_k N_l\,d\Omega \quad (6)$$

and where the components of the right-hand side vector b are given by

$$\boldsymbol{b}_k=\int_\Omega \hat{J}_s N_k\,d\Omega. \quad (7)$$

From (5) and (6) follows that $\boldsymbol{A}$ is a sparse and complex symmetric matrix, i.e., $\boldsymbol{A}=\boldsymbol{A}^T$.

This paper deals with solving (3) efficiently. Motivated by previous experience [3], this will be performed by using an *algebraic multigrid* solver. Algebraic multigrid solvers were orginally developed to treat symmetric positive definite problems. The purpose of this work is to consider its use to solve complex valued problems.

III. Geometric Multigrid

Multigrid methods for solving partial differential equations [4]–[6] are iterative methods that combine discretizations on grids of varying mesh density. High frequency error components are damped on a fine grid, whereas low frequency error components are transferred to the next coarser grid. On this next coarser grid, low frequency components appear as high frequent ones again, and the multigrid idea can be applied recursively. This recursion terminates when the cost of solving the linear system on the coarse grid becomes negligible.

To describe the two grid method formally, let $\boldsymbol{A}_h\boldsymbol{x}_h=\boldsymbol{b}_h$ and $\boldsymbol{A}_H\boldsymbol{x}_H=\boldsymbol{b}_H$ be a fine and coarse grid discretization of (1), respectively. Given an approximation $\boldsymbol{x}_h^k$ to the fine grid

equations, applying of few (typically one or two) steps of an stationary iterative scheme

$$\boldsymbol{x}_h^k \rightarrow \overline{\boldsymbol{x}}_h^k \quad \text{where } \overline{\boldsymbol{x}}_h^k = \boldsymbol{x}_h^k - \boldsymbol{S}_h^{-1}(\boldsymbol{A}_h \boldsymbol{x}_h^k - \boldsymbol{b}_h) \tag{8}$$

like damped Jacobi ($\boldsymbol{S}_h$ is the scaled diagonal of $\boldsymbol{A}_h$) or Gauss–Seidel ($\boldsymbol{S}_h$ is the lower triangular part of $\boldsymbol{A}_h$) damps high frequency components in the error $\boldsymbol{e}_h^k = \boldsymbol{x}_h - \boldsymbol{x}_h^k$. Low frequency error components are eliminated by restricting the residual $\boldsymbol{r}_h^k = \boldsymbol{b}_h - \boldsymbol{A}_h \overline{\boldsymbol{x}}_h^k$ to the coarser grid by the restriction operator I_h^H, solving the defect equation

$$\boldsymbol{A}_H \boldsymbol{e}_H^k = I_h^H \boldsymbol{r}_h^k \tag{9}$$

on the coarser grid, interpolating the resulting correction using the interpolation operator I_H^h back to the fine grid and adding it to the existing approximation $\overline{\boldsymbol{x}}_h^k$

$$\begin{aligned} \overline{\boldsymbol{x}}_h^k \rightarrow \overline{\overline{\boldsymbol{x}}}_h^k \quad \text{where } \overline{\overline{\boldsymbol{x}}}_h^k &= \overline{\boldsymbol{x}}_h^k + I_H^h \boldsymbol{e}_h^k \\ &= \overline{\boldsymbol{x}}_h^k + I_H^h \boldsymbol{A}_H^{-1} I_h^H \boldsymbol{r}_h^k \\ &= \overline{\boldsymbol{x}}_h^k + I_H^h \boldsymbol{A}_H^{-1} I_h^H (\boldsymbol{b}_h - \boldsymbol{A}_h \overline{\boldsymbol{x}}_h^k). \end{aligned} \tag{10}$$

As the last operation introduces high frequency components again, a few post smoothing steps may be performed.

Multigrid methods are known to be *optimal* in the sense that the number of iterations required to reach a fixed accuracy is independent of the mesh size. Krylov subspace methods can be applied to accelerate multigrid techniques. In such cases multigrid acts as a preconditioner. Using multigrid in this way also tends to stabilize the convergence behavior.

IV. Algebraic Multigrid

Electromagnetic problems of technical importance are often posed on complex geometries. The coefficients in the governing partial differential equations may be strongly varrying or have large discontinuities. Computational grids are therefore unstructured and refined to capture local effects if necessary. For such difficult problems efficient *geometric* multigrid methods are cumbersome to implement. It not immediately clear how a sequence of coarser meshes can be constructed, and which smoothers are appropriate for the given problem.

Algebraic multigrid (AMG) methods offer a solution to this problem by providing the advantages of geometric multigrid techniques in a black box solver. This black box solver requires as sole input the linear system to be solved. In the literature several approaches to AMG can be found (see [7] and references therein). In the following we will describe the Brandt–Ruge–Stüben approach to AMG for real symmetric positive definite problems [8].

The AMG solution process can be divided in two phases. In a setup phase, the algorithm constructs fully automically (i.e., without user intervention) a hierarchy of coarser meshes and the corresponding linear systems. To do so, the algorithm extracts from the fine grid matrix information about the strength of coupling between different nodes. In the cycling phase, this hierarchy of discrete problems is used to solve the problem by usual multigrid cycling.

When constructing the hierarchy of coarser discretizations, algebraic multigrid tries to balance the quality of the smoother and the coarse grid correction. Error components not damped by the coarse grid correction must be taken care of by the smoother and vice versa. The smoother in AMG is a simple point Gauss–Seidel smoother and a major part of the work is invested in building a coarse grid correction that makes up for the simplicity of the smoother. The coarser grid equivalent $\boldsymbol{A}_H$ of the system matrix $\boldsymbol{A}_h$ is built by using the Galerkin formula

$$\boldsymbol{A}_H = I_h^H \boldsymbol{A}_h I_H^h. \tag{11}$$

For symmetric problems the restriction operator is chosen to be the transpose of the prolongation operator: $I_h^H = (I_h^H)^T$. Given the smoother, the restriction and the coarser level discretization, only the selection of coarse grid points and the construction of the interpolation operator remains to be detailed.

The coarse grid selection induces a partition of the fine grid variables Ω^h into two disjoint sets $\Omega^h = C^h \cup F^h$, with C^h and F^h the coarse grid and fine grid variables respectively. The next coarse grid Ω^H is then identified with C^h. Each fine grid point $i \in F^h$ is interpolated from a subset $P_i^h \subset C^h$ of the coarse grid variables, called the *interpolatory variables* to point i. The interpolation operator has the form:

$$e_i^h = (I_H^h e^H)_i = \begin{cases} e_i^H & \text{if } i \in C^h \\ \displaystyle\sum_{k \in P_i^h} w_{ik}^h e_k^H & \text{if } i \in F^h. \end{cases} \tag{12}$$

The interpolation is constructed by requiring that smooth errors are accurately transferred from coarse to fine grids. The concept of smoothness can be defined purely algebraically: *algebraically* smooth errors satisfy

$$\boldsymbol{A}_{ii} e_i + \sum_{j \in N_i} \boldsymbol{A}_{ij} e_j \approx 0, \tag{13}$$

where $N_i = \{j \neq i \colon \boldsymbol{A}_{ij} \neq 0\}$. This simple fact about smooth errors gives the basic information from which the interpolation operators are deduced. The construction of interpolation based solely on (13) results in an interpolation for which the size of the sets of interpolation variables P_i^h, $i \in F^h$, is too large, resulting in computationally too expensive cycles. In truncating the size of these sets, each fine grid point has to remain sufficiently connected to its set of interpolatory coarse grid points. The truncation strategy leads to a heuristics to select the coarse grid. Once the coarse grid has been selected, the interpolation weights w_{ik}^h in (12) can be calculated.

V. Geometric Multigrid for the Complex Helmholtz Equation

By setting $\omega = 0$ in equation (1), the Poisson equation is obtained. For the latter type of equation, multigrid techniques are known to be efficient. To investigate the influence of the value of ω on the convergence behavior of geometric multigrid, we consider the following model differential problem on the square $\Omega = [0, L] \times [0, L]$,

$$-\frac{\partial^2 \hat{A}_z}{\partial x^2} - \frac{\partial^2 \hat{A}_z}{\partial y^2} + \boldsymbol{j}\omega\sigma\mu\hat{A}_z = \mu\hat{J}_s \tag{14}$$

where $\mu = 1/\nu$ is the permeability, subject to inhomogeneous Dirichlet boundary conditions on the whole boundary of Ω such that for $\omega \neq 0$ the function

$$\hat{A}_z = \frac{\hat{J}_s}{\boldsymbol{j}\omega\sigma}\left(1 - \frac{\cosh\left[\sqrt{\boldsymbol{j}\omega\sigma\mu}\,(x - L/2)\right]}{\cosh\left[\sqrt{\boldsymbol{j}\omega\sigma\mu}\,L/2\right]}\right) \tag{15}$$

TABLE I
NUMBER OF MULTIGRID CYCLES AS A FUNCTION OF MESH WIDTH FOR DIFFERENT FREQUENCIES

	Number of Cycles		
Mesh Width	f = 1e-5	f=50	f=5e4
4 x 4	11	11	9
8 x 8	12	12	12
16 x 16	13	12	13
32 x 32	12	12	18
64 x 64	12	12	29
128 x 128	12	12	14
256 x 256	12	12	12
512 x 512	12	12	12

is the exact solution to the problem. This solution is independent of y. We discretize the problem on a regular mesh with mesh-width $h = L/2^k$, $k = 2, 3, \cdots$ in both x and y direction using linear triangular finite elements. The various parameters have the following values: $L = 0.01$ m, $\sigma = 0.57 * 10^8\ \Omega^{-1}\ \mathrm{m}^{-1}$, $\mu = 4\pi * 10^{-7}\ \mathrm{TmA}^{-1}$, $J_s = 10^6\ \mathrm{Am}^{-2}$, and $\omega = 2\pi f$, where f is the frequency. The matrix of the resulting linear system can be represented by the stencil

$$\boldsymbol{A} \sim \frac{1}{h^2}\begin{bmatrix} 0 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 0 \end{bmatrix} + \boldsymbol{j}\,\frac{\omega\sigma\mu}{12}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 6 & 1 \\ 0 & 1 & 1 \end{bmatrix}. \quad (16)$$

We solve this linear system using geometric multigrid. We choose standard $h \to 2h$ coarsening, the W-cycle, seven-point restriction and interpolation and red-black Gauss–Seidel as smoother [6]. In Table I we listed the number of cycles required to reduce the initial residual by a fixed amount (namely by 10^{-12}) as a function of the mesh width for different frequencies. From Table I, we conclude that for a given frequency, the required number of cycles reaches an upper limit, indicating that the convergence is mesh size independent.

In Table II, we listed the asymptotic convergence factor ρ of the multigrid algorithm as a function of the frequency for various mesh widths. From Table II, we conclude that the speed of convergence decreases with increasing frequency untill the frequency reaches a threshold value. By increasing the frequency beyond this value, the multigrid algorithm speeds up again.

The fast convergence for high frequencies relative to the mesh width is irrelevant from a practical point of view as for such high frequencies a finer discretization is required to ensure the accuracy of the numerical solution. Furthermore, equation (1) is derived from the Maxwell equations assuming low frequency and thus neglecting the displacement current density.

VI. ALGEBRAIC MULTIGRID FOR COMPLEX SYMMETRIC SYSTEMS

To solve the complex linear system (3) by AMG, we rewrite the complex system as an equivalent real system of double dimension, and use a solver for *systems* of coupled PDE's. The development of AMG solvers for systems of PDE's is still a topic of ongoing research. Preliminary ideas can be found in [8]. In our experiments we use the code currently being developed by K. Stüben at GMD [7].

TABLE II
ASYMPTOTIC CONVERGENCE FACTOR AS A FUNCTION OF THE FREQUENCY FOR DIFFERENT MESH WIDTHS

Mesh Width 32 x 32								
f	1e-6	1	1e4	1e5	2e5	3e5	5e5	1e6
ρ	0.12	0.12	0.14	0.35	0.40	0.35	0.18	0.13

Mesh Width 128 x 128								
f	1e-6	1	1e4	2e5	1e6	2e6	4e6	5e6
ρ	0.12	0.12	0.125	0.13	0.35	0.37	0.35	0.30

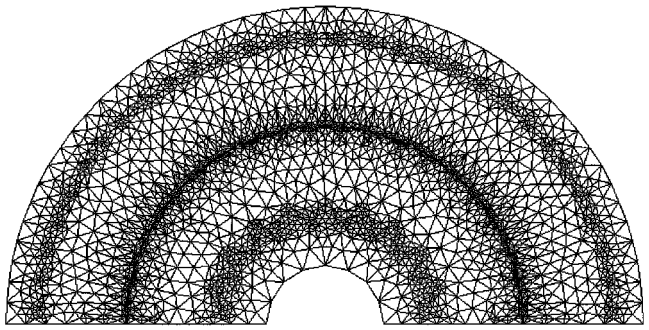

Fig. 1. Finite element mesh of an induction motor.

In rewriting the complex system as an equivalent real one, two alternatives exists: either a nonsymmetric variant

$$\begin{pmatrix} \boldsymbol{A}_R & -\boldsymbol{A}_I \\ \boldsymbol{A}_I & \boldsymbol{A}_R \end{pmatrix}\begin{pmatrix} \boldsymbol{x}_R \\ \boldsymbol{x}_I \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_R \\ \boldsymbol{b}_I \end{pmatrix}, \quad (17)$$

or an indefinite variant

$$\begin{pmatrix} \boldsymbol{A}_R & \boldsymbol{A}_I \\ \boldsymbol{A}_I & -\boldsymbol{A}_R \end{pmatrix}\begin{pmatrix} \boldsymbol{x}_R \\ -\boldsymbol{x}_I \end{pmatrix} = \begin{pmatrix} \boldsymbol{b}_R \\ \boldsymbol{b}_I \end{pmatrix}. \quad (18)$$

For our purposes, (17) has to be chosen as the code requires the diagonal of the system matrix to be positive.

The system-AMG code builds the coarser discretizations solely based on the diagonal blocks of the matrix (17), which, in our case, is just the real part of matrix (3). We use a V-cycle, with one pre and post smoothing step, and use the code as a preconditioner for BiCG-STAB [9].

VII. A PRACTICAL EXAMPLE

To compare the solution time of different solvers, a 400 kW induction machine is taken as an example. Thanks to symmetry, only two poles have to be modeled. The solid rotor bars are short circuited. The stator windings are excited by a three-phase alternating current system of 154 A. The finite element mesh and the magnetic flux lines are presented in Figs. 1 and 2 respectively. The final mesh was obtained after 4 adaptive refinement steps, and contains a total of 151 504 elements.

We compare the AMG/BiCG-STAB solver with symmetric successive overrelaxation preconditioned Quasi Minimal Residual (QMR) and with incomplete LU (ILU) preconditioned Complex Orthogonal Conjugate Gradient (COCG) solvers [10]. The COCG solver was taken from the PETSc package [11], and before applying the ILU preconditioner, the matrix was reordered to reduce the bandwidth. In Fig. 3, we

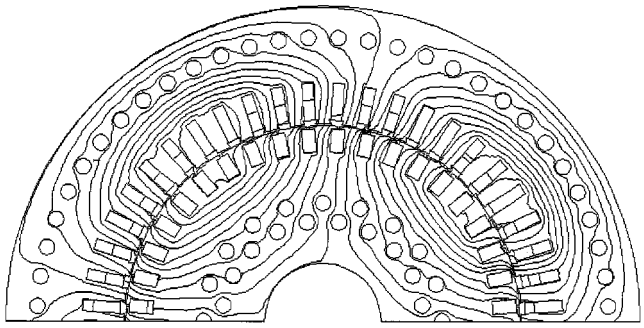

Fig. 2. Magnetic flux line plot of an induction motor.

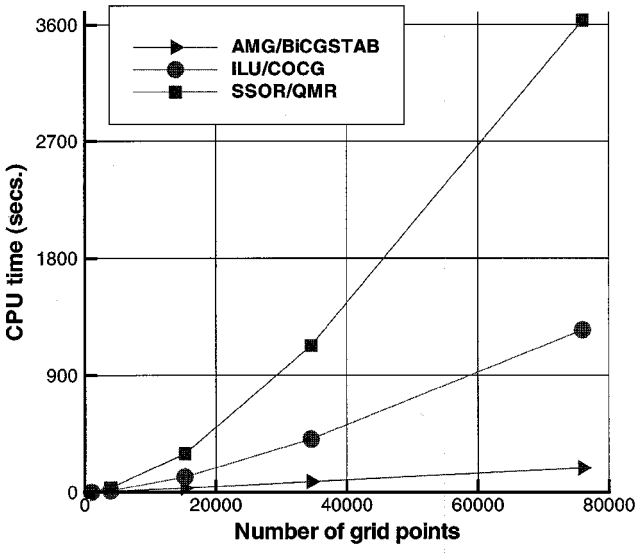

Fig. 3. Timings of different preconditioned iterative solvers.

plotted the CPU-time required by the different solvers to solve the linear system at each of the four adaptation steps.

VIII. Conclusions

We presented the application of an algebraic multigrid code for systems of coupled PDE's to solve linear systems resulting from the finite element discretization of quasistatic time-harmonic Maxwell equations. In the linear calculation of an induction motor the use of AMG as a precondioner resulted in a six-fold reduction of the CPU time compared to an optimized incomplete LU factorization and in a twenty-fold reduction compared to symmetric successive overrelaxation.

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