Solution Strategies for Transient, Field-Circuit Coupled Systems

Herbert De Gersem, Ronny Mertens, Domenico Lahaye, Stefan Vandewalle, and Kay Hameyer

***Abstract*—Transient simulation time for field-circuit coupled models of realistic electromagnetic devices becomes unacceptably high. A magnetodynamic formulation is coupled to an electric circuit analysis, yielding a sparse, symmetric and indefinite matrix. The unknown circuit currents correspond to negative eigenvalues in the matrix spectrum. The Quasi-Minimal Residual method performs better than the Minimal Residual approach that is restricted to positive definite preconditioners. The positive definite variant is solved by the Conjugate Gradient method without explicitly building the dense coupled matrix. As an example, both approaches are applied to an induction motor.**

***Index Terms*—Electromagnetic coupling, finite element methods, induction motors, iterative methods.**

I. Introduction

FINITE element simulation techniques are commonly used in the design and optimization of electromagnetic devices. The computation of the dynamic behavior of magnetic fields involves the simulation of the electric network that excites or is excited by the magnetic field. As the differential equations representing both phenomena are linearly dependent upon each other, simulation by means of one coupled system matrix is particularly attractive [1]. For a large range of technical devices operating at low frequencies, a clear distinction can be made between electrically conducting and nonconducting media. As a consequence, a description of the electric behavior of the device in terms of a lumped parameter model may reach a sufficient accuracy. The relative difference in permeability, however, is much lower. Moreover, the permeability may be nonlinear and the paths followed by the magnetic flux are usually rather irregular. As a consequence, the magnetic model requires a finer discretization, e.g., by means of finite elements.

Both the coupling of two physical phenomena and the hybrid nature of the discretization methods assign specific properties to the coupled system matrix. Magnetodynamic models of transformers, induction machines and induction furnaces are relatively small but have to be simulated many times. In the case of transient simulation, a huge number of sequential solutions is required. This fact justifies a detailed study of the influence of the field-circuit coupling mechanism on the efficiency of the system solution. In this paper, coupling schemes are selected from the viewpoint of the resulting system properties. Appropriate iterative solving techniques for the coupled system matrix are developed. The coupling techniques are judged upon the efficiency of the applicable iterative solvers.

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II. Magnetic Finite Element Model

A 2D quasistatic magnetic model is described by the magnetodynamic equations

$$-\frac{\partial}{\partial x}\left(\nu\,\frac{\partial A_z}{\partial x}\right)-\frac{\partial}{\partial y}\left(\nu\,\frac{\partial A_z}{\partial y}\right)+\sigma\,\frac{\partial A_z}{\partial t}=\frac{\sigma}{\ell}\,V_{sol}, \tag{1}$$

$$-\frac{\partial}{\partial x}\left(\nu\,\frac{\partial A_z}{\partial x}\right)-\frac{\partial}{\partial y}\left(\nu\,\frac{\partial A_z}{\partial y}\right)=\frac{N_t}{\Delta_{str}}\,I_{str}, \tag{2}$$

$$I_{sol}=G_{sol}V_{sol}-\int_{\Omega_{sol}}\sigma\,\frac{\partial A_z}{\partial t}\,d\Omega, \tag{3}$$

$$V_{str}=R_{str}I_{str}+\frac{N_t\ell}{\Delta_{str}}\int_{\Omega_{str}}\frac{\partial A_z}{\partial t}\,d\Omega. \tag{4}$$

A_z is the z-component of the magnetic vector potential. ν and σ are the reluctivity and the conductivity. ℓ is the length of the 2D model. A solid conductor is described by (1), (2) and (3), (4) as a function of the voltage V_{sol}, the current I_{sol} and the admittance G_{sol}. Eddy currents in the stranded conductor model are neglected. A stranded conductor with N_t turns and cross-section Δ_{str}, is described by (1), (2) and (4) as a function of the current I_{str}, the voltage V_{str} and the resistance R_{str}. Ω_{sol} and Ω_{str} are the domains in the 2D model corresponding to the solid conductor and the stranded conductor respectively.

For space discretization, linear triangular finite elements are used. For time discretization, the Galerkin time-stepping scheme ($\alpha = 2/3$) with fixed time step Δt is applied. The mechanical displacement is considered by a moving band technique [2].

III. Electric Circuit Coupling

The field-circuit coupling, applied here, is the hybrid analysis method described in [3] and [4]. As the method allows both unknown currents and unknown voltages to appear in the model, the formulation can serve as a reference from which more specific approaches can be derived. A tree, traced through the circuit, divides the circuit into two sets of branches.

The tree branches correspond to the admittance matrix $\boldsymbol{G}_T$, the unknown voltages $\boldsymbol{v}_T$ and their associated cutset equations. The links correspond to the impedance matrix $\boldsymbol{R}_L$, the unknown currents $\boldsymbol{i}_L$ and their associated loop equations. The coupled system is

$$\begin{bmatrix} \alpha\boldsymbol{K}+\dfrac{\boldsymbol{R}}{\Delta t} & \alpha\boldsymbol{Q}_T & -\alpha\boldsymbol{P}_L \\ \alpha\boldsymbol{Q}_T^T & \chi\boldsymbol{G}_T & \chi\boldsymbol{D}_{T,L} \\ -\alpha\boldsymbol{P}_L^T & -\chi\boldsymbol{B}_{L,T} & -\chi\boldsymbol{R}_L \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{v}_T \\ \boldsymbol{i}_L \end{bmatrix} = \begin{bmatrix} \boldsymbol{T}_1 \\ \boldsymbol{T}_2 \\ \boldsymbol{T}_3 \end{bmatrix}. \quad (5)$$

$\boldsymbol{K}$, $\boldsymbol{R}$, $\boldsymbol{Q}_T$ and $\boldsymbol{P}_L$ follow from discretizing (1)–(4) [3]. $\boldsymbol{D}_{T,L}$ and $\boldsymbol{B}_{L,T}$ are the fundamental cutset and loop matrices associated with the tree. The righthandside of (5) depends on the known voltage and current sources and the solution at the previous time step. Symmetry is preserved by the factor $\chi = \alpha\Delta t/\ell$ and the property $\boldsymbol{B}_{L,T} = -\boldsymbol{D}_{T,L}^T$ [5].

Field-circuit couplings are required to be reliable and applicable to arbitrary connected circuits. In the hybrid approach applied here, the difficulties related to the particular connections of stranded conductors, solid conductors, capacitors and inductors are resolved by the tree tracing procedure. Optional desired properties of the resulting coupled system are symmetry and sparsity. In this paper, the impact of these properties on the efficiency of the iterative system solution, is examined.

From (5), some common coupling approaches may be derived. The elimination of all $\boldsymbol{v}_T$ corresponding to solid conductors, yields the loop current formulation described in [1] and [6]. The elimination of all currents that are not related to stranded conductor links, together with the transformation from branch voltages to nodal voltages, leads to the popular nodal analysis method presented in [7]. These approaches combine currents and voltages, the one as principal circuit unknowns, the other whenever indispensable to retain the sparsity. The further elimination of stranded conductor link currents [7] or solid conductor voltages [8] yields a pure nodal analysis or a pure loop current analysis but spoils the sparsity of the finite element equations.

IV. System Properties

The coupled system matrix consists of n_{FEM} finite element equations related to the n_{FEM} nodes in the FE mesh, n_{tw} cutset equations related to n_{tw} unknown tree branch voltages and n_{ln} loop equations related to n_{ln} unknown loop currents. The spectrum of the coupled system matrix of a benchmark model is presented in Fig. 1. The finite element diagonal block $\alpha\boldsymbol{K}+\boldsymbol{R}/\Delta t$ is related to the parabolic and elliptic equations (1)–(2) and is positive definite. Both immittance matrices are diagonal. For the most general case described in [3], it is easily shown that the transformed immittance matrices remain positive definite. The physical duality of currents with respect to voltages and magnetic vector potentials, appears in the matrix as indefiniteness caused by the negative definite diagonal block $-\boldsymbol{R}_L$. An appropriate congruence transform and Sylvester's law of inertia reveal that the number of negative eigenvalues equals the number of loop equations (e.g., 1 in Fig. 1) [9]. Similar *mixed* formulations appear in other disciplines, e.g. the numerical solution of the Stokes problem [10] and mixed formulations for magnetostatics [11]. The finite element method applied here, is also a *hybrid* method as it involves the simultaneous approximation of a field defined on the finite element mesh, voltages defined across the fundamental cutsets and currents defined in the fundamental loops. The matrix may be ill-scaled due to relative differences in material properties and discretization sizes.

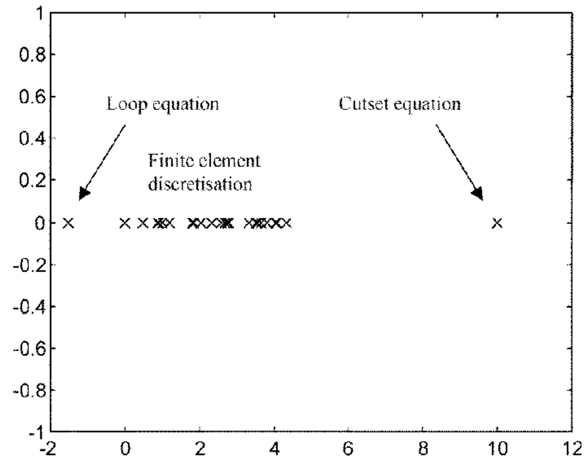

Fig. 1. Spectrum of a benchmark system matrix.

V. Krylov Subspace Acceleration

Large sparse systems are commonly solved by Krylov subspace iterative methods [12]. These methods search for an approximate solution of the system in a Krylov subspace of increasing dimension. Here, the solution procedure benefits from the symmetry of the system. For symmetric systems, a base for the Krylov space can be constructed by the Lanczos procedure. In this procedure, the orthogonalization of a new vector with respect to the current base consists of a recurrence relation only involving the three most recently obtained base vectors. The orthogonalization in the Arnoldi procedure suited for nonsymmetric matrices, has to be performed with respect to all previous base vectors. This requires all base vectors to be stored in memory and yields a growing computational cost per iteration step.

Krylov subspace solvers for symmetric, indefinite systems are the Minimal Residual (MINRES) method [13] and recently a variant of the Quasi Minimal Residual (QMR) method [14].

VI. Preconditioning

Preconditioning is recommended as ill-conditioned problems turn out to converge slowly [12]. MINRES, however, is restricted to positive definite preconditioners. As indefinite preconditioning is expected to establish a better convergence, MINRES is replaced by QMR. Common preconditioning techniques such as Jacobi, Gauss–Seidel, and Symmetric Successive Overrelaxation (SSOR), are extended to their block variants. Within each block, a preconditioner tuned to the corresponding part of the problem can be applied. A Jacobi block preconditioned system looks like

$$\begin{bmatrix} \boldsymbol{P} & 0 \\ 0 & \boldsymbol{C} \end{bmatrix}^{-1} \begin{bmatrix} \alpha\boldsymbol{K}+\dfrac{\boldsymbol{R}}{\Delta t} & \boldsymbol{B}^T \\ \boldsymbol{B} & \boldsymbol{C} \end{bmatrix} \begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{w} \end{bmatrix} = \begin{bmatrix} \boldsymbol{P} & 0 \\ 0 & \boldsymbol{C} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{T}_1 \\ \boldsymbol{T}_4 \end{bmatrix} \quad (6)$$

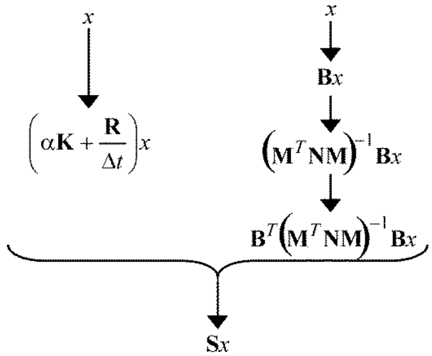

Fig. 2. Implicit multiplication procedure for $\boldsymbol{Sx}$.

with $\boldsymbol{P}$ an appropriate preconditioner for the finite element matrix part,

$$\boldsymbol{C} = \begin{bmatrix} \chi \boldsymbol{G}_T & \chi \boldsymbol{D}_{T,L} \\ -\chi \boldsymbol{B}_{L,T} & -\chi \boldsymbol{R}_L \end{bmatrix}, \tag{7}$$

$$\boldsymbol{B} = \begin{bmatrix} \alpha \boldsymbol{Q}_T^T \\ -\alpha \boldsymbol{P}_L^T \end{bmatrix}, \tag{8}$$

$$\boldsymbol{T}_4 = \begin{bmatrix} \boldsymbol{T}_2 \\ \boldsymbol{T}_3 \end{bmatrix} \tag{9}$$

and

$$\boldsymbol{w} = \begin{bmatrix} \boldsymbol{v}_T \\ \boldsymbol{i}_L \end{bmatrix}. \tag{10}$$

VII. Positive Definite Alternative

The elimination of $\boldsymbol{i}_L$ in (5) yields a positive definite system matrix, equivalent to the nodal circuit analysis presented in [7]. The explicit substitution of the last row of (5) in $\alpha \boldsymbol{K} + \boldsymbol{R}/\Delta t$ would create a dense Schur complement. For the numerical example, described below, the coupled system (5) has 47 000 nonzero elements whereas the explicit Schur complement contains 1 989 732 nonzeros. This would destroy the efficiency of the matrix-vector product in the iterative method.

Here, a Schur complement of the whole circuit part is constructed.

$$\underbrace{\left(\alpha \boldsymbol{K} + \frac{\boldsymbol{R}}{\Delta t} - \boldsymbol{B}^T \boldsymbol{C}^{-1} \boldsymbol{B}\right)}_{\boldsymbol{S}} \boldsymbol{A} = \boldsymbol{T}_1 - \boldsymbol{B}^T \boldsymbol{C}^{-1}.\boldsymbol{T}_4. \tag{11}$$

The positive definite system is solved by the Conjugate Gradient (CG) method. It is possible to design an implicit multiplication procedure for $\boldsymbol{Sx}$ (Fig. 2). $\boldsymbol{C}$ is factorized in advance:

$$\boldsymbol{C} = \boldsymbol{M}^T \boldsymbol{N} \boldsymbol{M} \tag{12}$$

with $\boldsymbol{M}$ an upper triangular and $\boldsymbol{N}$ a diagonal matrix. The corresponding computational cost is negligible because $\boldsymbol{C}$ contains typically only a few hundred equations. This implicit approach enables the application of the underlying positive definite system without requiring the explicit construction of the matrix.

A difficulty of this approach is the choice of an appropriate preconditioner for $\boldsymbol{S}$. The proper matrix is not available and the application of Jacobi, Gauss–Seidel, or SSOR preconditioning would require a similar implicit approach. Another possibility is applying a good preconditioner for $\alpha \boldsymbol{K} + \boldsymbol{R}/\Delta t$ as a preconditioner for $\boldsymbol{S}$. Then, the solution process may benefit from an available powerful preconditioning technique for parabolic partial differential equations, such as Incomplete Cholesky or Algebraic Multigrid (AMG) [15]. Although, it should be mentioned that this preconditioner does not count for the electric behavior of the system. As a consequence, the efficiency of this approach has to be proved experimentally for each particular model under consideration.

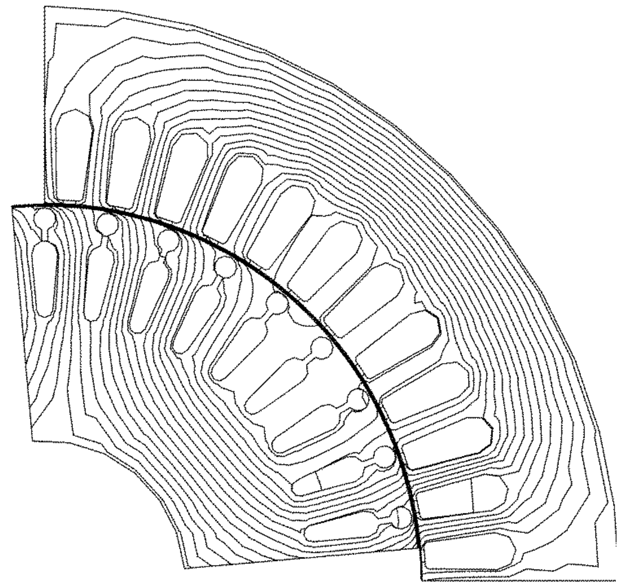

Fig. 3. Magnetic flux lines of the induction motor.

VIII. Application

The geometry of a four-pole 45 kW induction motor is discretized by 6010 elements (Fig. 3). The topological circuit treatment yields 314 extra unknown voltages, 40 extra unknown currents and their corresponding equations in the coupled system. In the first numerical experiment, QMR featuring the indefinite SSOR preconditioner is compared to MINRES with a corresponding definite preconditioner, denoted by "| SSOR|." The SSOR preconditioner applied to QMR, factorized as $\boldsymbol{L}^T \boldsymbol{D} \boldsymbol{L}$ is adapted to MINRES as $\boldsymbol{L}^T \boldsymbol{E} \boldsymbol{L}$ where the diagonal elements of $\boldsymbol{E}$ are the absolute values of the diagonals of $\boldsymbol{D}$. For the models simulated here, the possibility to apply indefinite preconditioning is more important than the true minimization property of MINRES (Fig. 4). The effect of Jacobi block preconditioning, denoted by "JAC(*, *)," is established in Table I. It can be concluded that the presence of the electromagnetic coupling terms in the preconditioner, has a substantial influence on the convergence behavior of the Krylov subspace solvers. The Generalized Minimal Residual (GMRES) method, relying upon the Arnoldi procedure, is used to demonstrate the importance of matrix symmetry of the coupled system for the efficiency of the iterative solution. In the third experiment, the positive definite Schur complement is solved. The efficiency of several preconditioners for CG is examined in Fig. 5 and Table I. AMGCG is

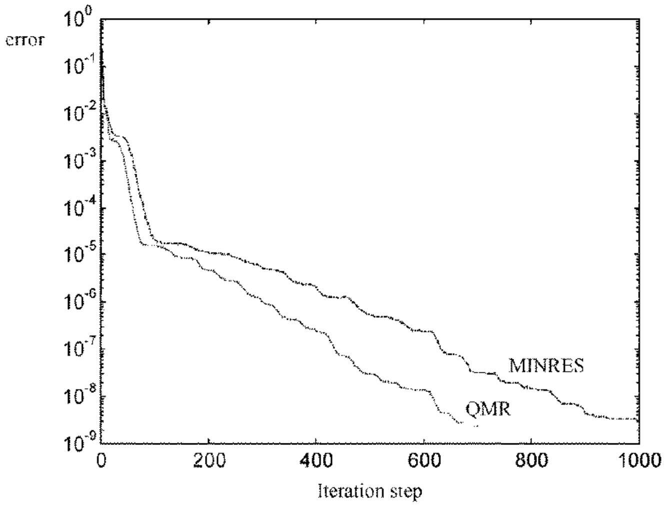

Fig. 4. Convergence of QMR with an indefinite preconditioner compared to MINRES with a positive definite preconditioner.

TABLE I
ITERATION COUNTS AND COMPUTATION TIMES OF THE ITERATIVE SYSTEM SOLUTION FOR 1 TIME STEP OF THE TRANSIENT SIMULATION

	iteration steps	computation time [s]
MINRES	1997	59.80
QMR	1330	40.23
GMRES	1031	159.71
|SSOR|MINRES	476	11.03
SSORQMR	320	7.41
SSORGMRES	312	38.13
JAC(SSOR,LU[b])GMRES	325	53.91
JAC(AMG,LU[b])QMR	245	5.23
CG[a]	2376	30.06
SSORCG[a]	1105	17.32
AMGCG[a]	199	9.28

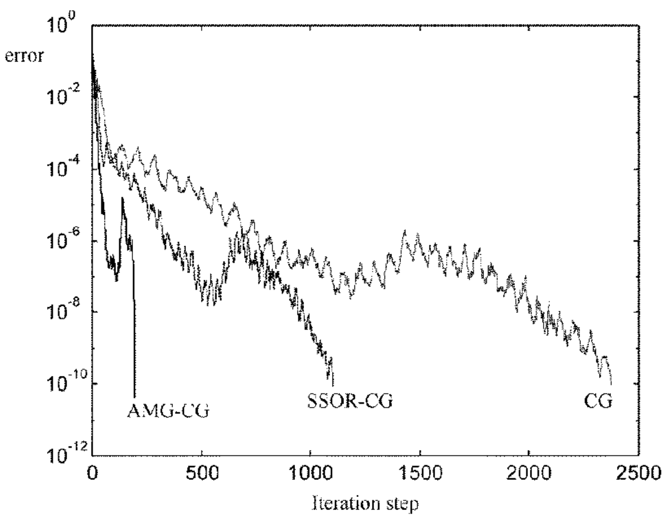

Fig. 5. Convergence of CG applied to the Schur complement preconditioned by SSOR or AMG.

promising compared to all other approaches. Better convergence is expected if the AMG technique is extended to incorporate the circuit couplings.

IX. CONCLUSIONS

The properties of the hybrid system matrix of a transient field- circuit coupling model, are studied. The Quasi-Minimal Residual method, solving the sparse, symmetric and indefinite system, is suited for indefinite preconditioning and establishes a better convergence when compared to the Minimal Residual method. Block preconditioning enables the application of preconditioning techniques for the partial problems. The positive definite alternative formulation is implicitly built and solved by the Conjugate Gradient method. The techniques developed here, increase the speed and the reliability of transient simulations, here as an example applied to an induction motor. The numerical results reveal that preserving the symmetry of the system is very important. It is not required that the coupled system remains positive definite.

REFERENCES

[1] I. A. Tsukerman, A. Konrad, G. Meunier, and J. C. Sabonnadière, "Coupled field-circuit problems: Trends and accomplishments," *IEEE Trans. Magn.*, vol. 29, pp. 1701–1704, Mar. 1993.

[2] N. Sadowski, Y. Lefèvre, M. Lajoie-Mazenc, and J. Cros, "Finite element torque calculation in electrical machines while considering the movement," *IEEE Trans. Magn.*, vol. 28, no. 2, pp. 1410–1413, Mar. 1992.

[3] H. De Gersem, R. Mertens, U. Pahner, R. Belmans, and K. Hameyer, "A topological method used for field-circuit coupling," *IEEE Trans. Magn.*, vol. 34, pp. 3190–3193, Sept. 1998.

[4] R. Mertens, H. De Gersem, and K. Hameyer, "Transient field-circuit coupling based on a topological approach," in *9th International Symposium on Electromagnetic Fields-ISEF'99*, Pavia, Italy, Sept. 23–25, 1999, pp. 17–23.

[5] L. O. Chua and P. M. Lin, *Computer Aided Analysis of Electronic Circuits—Algorithms and Computational Techniques*, NJ: Prentice-Hall, 1975.

[6] I. A. Tsukerman, A. Konrad, and J. D. Lavers, "A method for circuit connections in time-dependent eddy current problems," *IEEE Trans. Magn.*, vol. 28, pp. 1299–1302, Mar. 1992.

[7] J. Gyselinck and J. Melkebeek, "Numerical methods for time stepping coupled field-circuit systems," in *Proc. ELECTRIMACS'96*, Saint-Nazaire, 1996, pp. 227–232.

[8] Th. Dreher and G. Meunier, "3D modeling of electromagnets fed by alternating voltage sources," *IEEE Trans. Magn.*, vol. 29, pp. 1341–1344, Mar. 1993.

[9] G. H. Golub and C. F. Van Loan, *Matrix Computations*. Baltimore: The John Hopkins University Press, 1989.

[10] H. C. Elman, D. J. Silvester, and A. J. Wathen, "Iterative Methods for Problems in Computational Fluid Dynamics," Oxford University Computing Laboratory, Technical Report NA-96/19, 1996.

[11] I. Perugia, "A field-based mixed formulation for the 2-D magnetostatic problem," *SIAM J. Numer. Anal.*, vol. 34, pp. 2382–2391, 1997.

[12] Y. Saad, *Iterative Methods for Solving Linear Systems*. Boston: PWS Publishing Company, 1996.

[13] C. C. Paige and M. A. Saunders, "Solution of sparse indefinite systems of linear equations," *SIAM J. Numer. Anal.*, vol. 12, pp. 617–629, 1975.

[14] R. W. Freund and N. M. Nachtigal, "A new Krylov-subspace method for symmetric indefinite linear systems," in *Proc. of the 14th IMACS World Congress on Comp. and Appl. Math.*, W. F. Ames, Ed: IMACS, 1994, pp. 1253–1256.

[15] J. Ruge and K. Stueben, "Algebraic multigrid," in *Multigrid Methods*. ser. Frontiers in Applied Mathematics, S. McCormick, Ed. Philadelphia, PA: SIAM, 1987, vol. 3, pp. 73–130.