em 3. *The divergence free wavelet bases Ψ^{df} fulfill Assumption 1 for n 3.*

Condition (a) is trivially fulfilled since the bilinear form $a(\cdot,\cdot)$ is ellip-
ıll of $H_0^1(\Omega)^n$, [10]. Since V is a closed subset of $H_0^1(\Omega)^n$, the norm
ence (14) for $s = 1$ already ensures (b) in Assumption 1. Finally, due
properties of $S(\lambda)$ and Ψ in (11), we obtain

$$a(\psi_\lambda^{df}, \psi_{\lambda'}^{df}) = \sum_{\mu \in S(\lambda)} \sum_{\mu' \in S(\lambda')} d_{\lambda,\mu} d_{\lambda',\mu'} a(\psi_\mu, \psi_{\mu'}) \ .$$

of (12) and the properties of $S(\lambda)$, we obtain that Ψ^{df} enforces
ous decay properties as Ψ which proves (c). □

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A Finite-Element/Equivalent-Circuit Two-Level Method for Magnetic Field Simulations

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Abstract A magnetic equivalent circuit serves as coarse representation within a multilevel solver for finite element magnetic simulation. The prolongation is based on the physical relation between the magnetic fluxes of the circuit and the magnetic vector potentials of the finite element problem. The multilevel technique is applied as a solver and as a preconditioner for the Conjugate Gradient algorithm. For the technical devices considered here, a deflated Conjugate Gradient solver combining a standard preconditioner with the multilevel idea, provides the best convergence.

1 Introduction

Finite element (FE) simulations are a principal component in the design and optimisation procedure for electrical devices, e.g. transformers, induction machines and actuators. Inside such machines, a magnetic field is excited by an applied electric current. The discretisation of the models of such machines using FEs suffers from the complicated geometry of technical devices with winding slots, cooling channels and small air gaps (Fig. 1). The coarsest grid that is constructable in a geometrical way, already requires a considerable number of elements. An exact solve of the corresponding FE problem is very expensive. Hence, the size and the complexity of this coarse grid problem may adversely affect the efficiency of a geometric multigrid algorithm designed to solve the magnetic problem on a fine grid.

In this paper, a coarse representation of the magnetic field problem based on equivalent circuits is inserted within a multilevel iterative solver. The nature of the resulting multilevel algorithm is hybrid, i.e., a combination of a distributed parameter problem (the partial differential equation (PDE)) and a lumped parameter problem (the equivalent circuit), but remains geometric.

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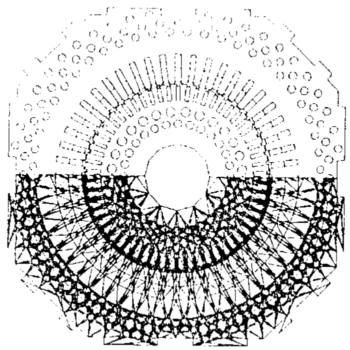


l. Geometry and mesh of a four-pole, three-phase induction motor model.

agnetic Finite Element Model

ations, considered here, consist of a subset of the Maxwell equations · with the constitutive relation between the magnetic flux density $\boldsymbol{B}$ magnetic field strength $\boldsymbol{H}$.

$$\nabla \cdot \boldsymbol{B} = 0 \tag{1}$$

$$\nabla \times \boldsymbol{H} = \boldsymbol{J} \tag{2}$$

$$\boldsymbol{B} = \mu \boldsymbol{H} \tag{3}$$

current density and μ the permeability. The continuity relation (1) is by the definition of the magnetic vector potential $\boldsymbol{A}$ by $\nabla \times \boldsymbol{A} = \boldsymbol{B}$. ns (2) and (3) are combined into a single equation

$$\nabla \times (\nu \nabla \times \boldsymbol{A}) = \boldsymbol{J} \tag{4}$$

reluctivity $\nu = 1/\mu$. Restricting (4) to two dimensions, we have that $_x, B_y, 0)$, $\boldsymbol{A} = (0, 0, A_z)$ and $\boldsymbol{J} = (0, 0, J_z)$, which yields a variable ontinuous coefficient diffusion equation:

$$-\frac{\partial}{\partial x}\left(\nu \frac{\partial A_z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\nu \frac{\partial A_z}{\partial y}\right) = J_z . \tag{5}$$

proximated by an expansion $\sum_{j=1}^{n} x_j N_j$ in terms of n linear triangu- asis functions N_j. The discrete form of (5) is given by $\mathbf{Ax} = \mathbf{b}$ with $_2 \nu \nabla N_i \cdot \nabla N_j d\Omega$ and $b_i = \int_\Omega J_z N_i d\Omega$. Matrix $\mathbf{A}$ is symmetric and definite.

ough the magnetic vector potential has no straightforward physical , it can be easily related to the magnetic flux. The magnetic flux ϕ a surface S is defined by

$$\phi = \int_S \boldsymbol{B} \cdot d\boldsymbol{S} = \oint_{\partial S} \boldsymbol{A} \cdot d\boldsymbol{\ell} . \tag{6}$$

which, in the 2D model considered here, is expressed as

$$\phi = \ell_d \left(A_{z2} - A_{z1}\right) . \tag{7}$$

ℓ_d is the length of the device in the z-direction. It is this relation that will be exploited in the construction of a hybrid FE/equivalent-circuit multilevel method.

3 Magnetic Equivalent Circuit Model

In the electrical engineering community, magnetic simulations are often performed by using MECs rather than the PDE formulation (5). The global computational domain is divided into subdomains, called ***flux tubes***. The magnetic flux ϕ through the tube and the magnetomotive force $\mathcal{V}_m$ across the tube are related to each other by Hopkinson's law, $\mathcal{V}_m = \mathcal{R}_m \phi$, with

$$\mathcal{R}_m = \int_{r_1}^{r_2} \frac{\nu(\boldsymbol{r})}{s(\boldsymbol{r})} d\boldsymbol{r}, \tag{8}$$

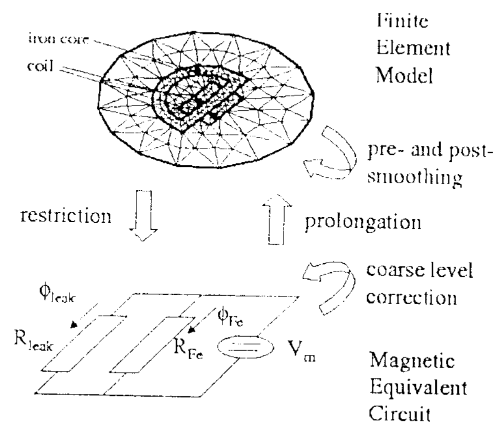
the magnetic reluctance. $\boldsymbol{r}$ is the running coordinate along the flux path and $s(\boldsymbol{r})$ is the cross-section of the flux tube. The magnetic reluctances are assigned as lumped parameters to the resistors in the equivalent circuit.

The MEC of a small benchmark inductor model (Fig. 2) consists of two magnetic reluctances: $\mathcal{R}_{Fe}$ representing the favourable iron flux path and $\mathcal{R}_{leak}$ accounting for the leakage flux through the surrounding air. The excitation winding operates as a magnetomotive source $\mathcal{V}_{ex} = N_t I$ with N_t the number of turns and I the excitation current. To solve the network, an unknown loop flux is assigned to each independent loop in the circuit (ϕ_{Fe} and ϕ_{leak} in Fig. 2). Kirchhoff's voltage law is applied to each independent loop:

$$\begin{pmatrix} \mathcal{R}_{Fe} & 0 \\ 0 & \mathcal{R}_{leak} \end{pmatrix} \begin{pmatrix} \phi_{Fe} \\ \phi_{leak} \end{pmatrix} = \begin{pmatrix} \mathcal{V}_{ex} \\ \mathcal{V}_{ex} \end{pmatrix} . \tag{9}$$

The small system matrix is positive definite.

The equivalent circuit approach for magnetics has certain advantages over the FE simulation technique. The equivalent circuit enables an easy derivation of technically relevant data and therefore provides a powerful interpretation technique to the design engineer. The equivalent circuit model can be highly accurate. State of the art circuit models for practical, standard devices attain accuracies for relevant quantities that are within the 1% range. In practice, a circuit model is always studied before proceeding to a FE model. Finally, one should also mention the small problem size related to the lumped parameter description as a major advantage. The major disadvantage for the circuit approach is the inability to consider local effects such as magnetic saturation, hysteresis and eddy currents. This problem, however, can partially be overcome by the introduction of nonlinear, lossy and frequency dependent lumped parameters into the circuit.



2. FEM/MEC two-level approach applied to a benchmark inductor model.

nite-Element/Equivalent-Circuit Two-Level ethod

ıpproach, we try to exploit the favourable properties of the MEC to the FE simulations of the PDE model. The FE and MEC modelling ıes are embedded within a two-level hierarchy (Fig. 2). The magnetic otential distributions corresponding to the unknown loop fluxes con- he prolongation from the MEC to the FE model. The prolongation are constructed on a heuristic basis. Each prolongation vector repre- homogeneous flux distribution in the parts of the model excited by esponding loop flux (Fig. 3). The vector potential varies linearly in s-section of the flux tube. The vector potential remains constant in ons that are not influenced by the considered loop flux. As restriction, int of the prolongation is preferred to the trivial injection defined by moothers, damped Jacobi and Gauss-Seidel relaxations are applied. coarse level, an exact solution of the MEC is performed.

Figure3. Prolongations of φ_{Fe} and φ_{leak}.

The multilevel method is symmetric and positive definite and can therefore serve as a preconditioner for the CG algorithm. The acceleration technique may improve the convergence behaviour of the overall iteration process. Numerical results will be given in Chap. 6.

5 Deflated Conjugate Gradient Method Combined with the Multilevel Strategy

Another complementary line of reasoning is set up from within the framework of Krylov subspace methods. The convergence of a Krylov subspace solver is determined by the spectral properties of the system matrix. For CG, the convergence is bound by

$$\left\|\mathbf{x}-\mathbf{x}^{(k)}\right\|_{\mathbf{A}} \leq 2\left\|\mathbf{x}-\mathbf{x}^{(0)}\right\|_{\mathbf{A}}\left(\frac{\sqrt{K}-1}{\sqrt{K}+1}\right)^{k} \tag{10}$$

with K the condition number and k the iteration step. Preconditioning by Jacobi (JAC), symmetric Gauss-Seidel (SGS) or symmetric successive overrelaxation (SSOR) improves the convergence substantially. The spectrum of the preconditioned matrix, however, still contains a few very small eigenvalues, yielding a large condition number and thus a poor convergence. As pointed out in [4], these eigenmodes are responsible for some stagnation points in the convergence history of CG. These eigenmodes seem to reflect the presence of regions with large relative differences in the material properties. This same observation is applied when the MEC model is built: the high permeable parts are distinguished from the low permeable parts of the model. As a consequence, the prolongation vectors constructed in Chap. 4 are related to the jumps in the material coefficients. They can serve as basis vectors of an approximative partial eigenspace $\mathbf{V}$ corresponding to the small eigenvalues left after standard preconditioning. A deflated version of the preconditioned CG algorithm is proposed in [4]. The projector

$$\mathbf{P}=\mathbf{I}-\mathbf{V}\mathbf{E}^{-1}(\mathbf{A}\mathbf{V})^{T} \tag{11}$$

with $\mathbf{E}=(\mathbf{A}\mathbf{V})^{T}\mathbf{V}$, projects a vector onto the space orthogonal to $\mathbf{V}$ with respect to the $(\ ,\ )_{\mathbf{A}}$ inner product. The solution process is split up into two complementary parts: a small problem solves for the contributions of the prolongation vectors to the solution:

$$\mathbf{x}_1=\mathbf{V}\mathbf{E}^{-1}\mathbf{V}^{T}\mathbf{b} \tag{12}$$

and a large problem, solved by CG,

$$\mathbf{P}^{T}\mathbf{A}\mathbf{x}_2=\mathbf{P}^{T}\mathbf{b}, \tag{13}$$

deals with the other components of the solution. Because $\mathbf{P}^{T}\mathbf{b}\in \mathrm{Ran}\,(\mathbf{P}^{T}\mathbf{A})$, CG is still applicable. The deflation removes the slowly converging modes out of the spectrum of the system matrix. The total solution is $\mathbf{x}=\mathbf{x}_1+\mathbf{P}\mathbf{x}_2$.

Application

wo-level FE/MEC method is applied as a stand-alone solver, as a pre-ioner for CG and as a simulation technique to approximate the partial pace used to deflate the preconditioned system. Two models are con-l: the benchmark inductor model of Fig. 2 and the induction machine 1. Thanks to the symmetry of the geometry and the excitation of the phase, four-pole induction machine, only half of the model has to be ered. The zero-load operation of the machine is the state in which the s rotating synchonously with respect to the rotating magnetic air gap xcited by the stator windings. In that case, no currents are induced rotor and an appropriate transformation of the model to a station-ordinate system enables the application of a magnetostatic solver. The es corresponding to the zero-load operation of the induction motor are l in Fig. 4.

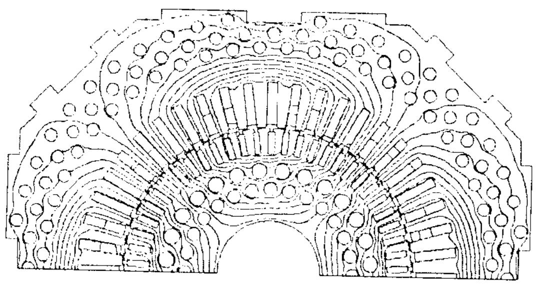


Figure4. Magnetic flux lines of the induction machine.

ition counts for the various methods are presented in Table 1. One cy-e two-level method consists of one forward Gauss-Seidel pre-smoothing e MEC correction and one backward Gauss-Seidel post-smoothing is observed that the two-level FEM/MEC approach performs better mpared to pure relaxation schemes. This novel method is however not tive to an algebraic multigrid (AMG) technique [3] for this elliptic . The speedup of the FEM/MEC two-level method with respect to relaxation schemes, is disappointing. The coarsening within the mul-pproach seems to be too aggressive for the simple smoothers applied

deflated CG algorithm benefits more from the coarse information by the hybrid multilevel iterative solver. To obtain the deflation each independent loop in the MEC is excited by a test flux. The tions of the corresponding MEC solutions constitute a set of linear lent deflation vectors. For the technical example, the application of ted CG algorithm combined with Incomplete Cholesky (IC) precon-

ditioning, results in a considerable speedup compared to the standard ICCG algorithm (Table 1).

Table1. Iteration counts: FEM/MEC compared to JAC, SGS, AMG and deflated CG with IC preconditioning (DICCG).

	Mod1	Mod1 + CG	Mod2 + CG
Size FE model	153	153	1951
Size MEC	2	2	53
JAC	768	27	2001
SGS	386	23	825
FEM/MEC + JAC	202	25	626
FEM/MEC + SGS	128	19	587
AMG	-	-	11
IC	-	12	383
DIC	-	9	62

7 Conclusions

The finite-element/equivalent-circuit multilevel method uses an additional level that is coarser than the smallest finite element mesh that can be constructed geometrically. Physical heuristics is applied to construct the prolongation operator. The method is applicable both as solver and as preconditioner. The multilevel approach is also combined with a standard preconditioner within a deflated Conjugate Gradient algorithm. The numerical tests applied to a technical model reveal a considerable convergence improvement using the hybrid two-level method.

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