FINITE ELEMENT STABILIZATION TECHNIQUES FOR CONVECTION-DIFFUSION PROBLEMS

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Abstract

In general, motion effects in a physical system are described by a convection-diffusion equation. If this equation is solved numerically by the Galerkin method, an unstable and oscillating solution is observed at higher speeds. A severely refined mesh can be used to avoid this phenomenon, but computation time might then become unacceptable. Moreover, since the exact solution of a high speed motion problem is characterized by one or more small and steep transitions, many nodes are located outside these interesting zones. Here, a more elegant way for solving convection-diffusion problems is presented.

1 Introduction

In electromagnetic problems, convection-diffusion equations describe systems with moving bodies, such as electrical motors or eddy current brakes. The two dimensional quasi-static magnetic convection-diffusion equation is given by [6]:

$$-\nabla \cdot \left(\nu_r \nabla A_z\right) + \mu_0 \sigma \vec{v} \cdot \nabla A_z = -\mu_0 \sigma \nabla V \; , \qquad (1)$$

where A_z is the z-component of the magnetic vector potential, V the electric scalar potential, $\vec{v}$ the velocity vector, ν_r the relative reluctivity, μ_0 the permeability of air and σ the electrical conductivity. The second order derivative with respect to the place, represents the diffusion term. The convection term is given by the first order derivative. When this equation is solved numerically on a pre-determined mesh by the Galerkin method, instabilities are found in the solution if the convection term dominates over the diffusion term [1,3]. Equation (1) reveals that this happens when the ratio

$$\frac{\sigma \|\vec{v}\|}{\nu_r} \qquad (2)$$

is too large. These oscillations do not have a physical interpretation. They are purely numerical.

2 Numerical problem

Consider the linear eddy current brake in figure 1. The iron rail moves to the right side. Only half of the model is shown because the brake is symmetric. The magnetic field, generated by the winding current, penetrates the rail and induces eddy currents there, due to the relative motion. The interaction between the magnetic field generated by the eddy currents and the main magnetic field generated by the winding current, causes a redistribution of the field towards the direction of the movement. Figure 2 shows a stable solution at relatively low speed and an unstable solution, observed at higher speeds.

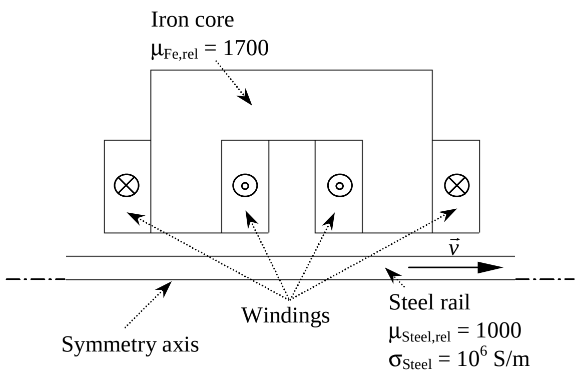


Figure 1: Configuration of a linear eddy current brake.

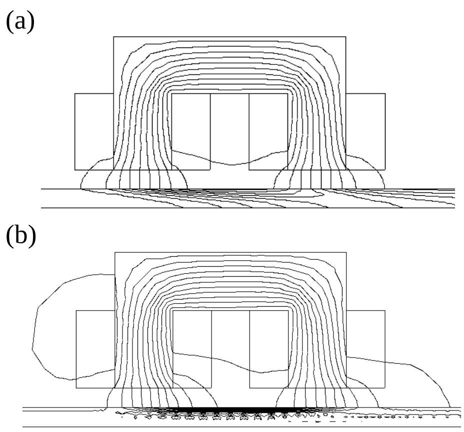


Figure 2: Galerkin solution, for (a) $\vec{v}$ = 2 m/s and (b) $\vec{v}$ = 150 m/s.

3 High speed solutions

For one dimensional problems, it is well known that oscillations occur when the dimensionless Peclet number is larger than one. The Peclet number is defined by [2,7]:

$$Pe = \frac{h\|\vec{v}\|\mu\sigma}{2} \quad , \tag{3}$$

where h is the characteristic mesh size. The permeability and the conductivity are determined by the materials. This implies that high speed problems are only solvable if very small elements are used. In practice, oscillations occur when the element size is relatively large compared to the distance over which the transitions take place. The Peclet number, as defined above, is only valid for one dimensional problems. However, for two or three dimensional problems, the same conclusion can be drawn.

To solve the motion problem at elevated velocities, and to avoid a strongly refined mesh in the entire moving region, the solution process is performed in two steps. In the first step, the location of the transitions is determined on a stable approximation for the solution. Therefore, two approaches are possible. Either the Galerkin method or the differential equation is altered. In the second step, the Galerkin method is applied to a mesh that is only severely refined in the transition zones.

3.1 Artificial diffusion

As numerical instabilities only occur in the solution when the convection term is large when compared to the diffusion term, it is sufficient to enlarge the contribution of the diffusion term artificially, in order to avoid the instabilities. Here, this is performed by increasing the value of the reluctivity in the moving region. This is equivalent to a reduction of the permeability. Reducing the permeability on its turn reduces the effect of the eddy currents and thus the interaction between the main field and the eddy current field. Therefore, this technique is called artificial diffusion (AD) [1,2,3].

Although the obtained solution is a numerical approximation for the solution of another differential equation, it still can resemble the solution of the basic equation. This is possible and feasible, because the main effect of artificial diffusion is smoothing the transitions. When the extra diffusion is higher, the difference with the exact solution is larger.

3.2. Streamline methods

Artificial diffusion simply introduces diffusion where it is required. The amount of diffusion depends on the material characteristics, the element size and the velocity. However, the method imposes the same diffusion in all directions. Artificial diffusion does not take advantage of the known direction of motion.

Each convection-diffusion problem can be considered in terms of its streamlines [4]. Streamlines are paths that imaginary points would follow in the velocity field. It can be shown that the transitions are steepest in the direction of the streamlines. In the crosswind direction, perpendicular to the streamlines, the transitions are smooth. More recent techniques, such as streamline diffusion (SD) and streamline upwind Petrov Galerkin (SUPG) consider this phenomenon [1,2,3].

In streamline diffusion, a term that only introduces diffusion in the direction of the streamlines, is added to the differential equation. This technique assumes that no smoothing is required in the crosswind direction. This extra term has the form:

$$\delta\nabla\cdot\left[\left(\vec{v}\cdot\nabla A\right)\vec{v}\right] \quad . \tag{4}$$

Now, the diffusion factor δ is the parameter influencing the stability of the solution. The new differential equation is solved by the Galerkin method. In the same way as in artificial diffusion, the numerical solution is an approximation for another, but similar problem. However, as diffusion is only introduced in the streamline direction, the problem is less altered when compared to the artificial diffusion approach.

SUPG does not change the differential equation. This method is characterized by the use of other weighting functions. The weighting functions are formed by a sum of a shape function and a function considering information about the streamlines. These weighting functions have the form:

$$W = N + \delta\vec{v}\cdot\nabla N \quad , \tag{6}$$

where N is a shape function. The first numerical solution is already an approximation of the exact solution, because no changes are made in the differential equation. The upwind factor δ stabilizes the solution.

4 Implementation

Here, a practical implementation of artificial diffusion is presented. The system matrix of a convection-diffusion problem is the sum of a diffusion matrix **K** and a motion matrix **M**. The contribution of the motion term on each element of the system matrix is either positive or negative. It has been observed that the solution remains stable, as long as the sign of all the elements in the system matrix **K**+**M** and the diffusion **K** matrix are equal. Therefore, to obtain a stable solution for a high speed problem, the entire diffusion matrix is multiplied by a factor f ($f > 1$), until the signs of all elements of the matrix f**K**+**M** are equal to the signs of **K**. This scheme is an artificial diffusion scheme, because every element of the diffusion matrix is proportional to the reluctivity of the material [5]. To improve the method, the multiplication is performed per finite element. The problem is minimally altered in this case. This yields the following implementation scheme:

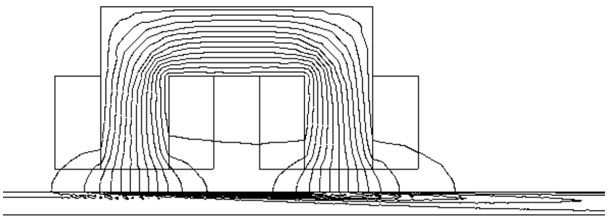
- Calculate the element diffusion matrix $\mathbf{K}^{(e)}$.
- Calculate the element motion matrix $\mathbf{M}^{(e)}$.
- Calculate the value $f^{(e)}$, such that

$$
\begin{aligned}
& sign\left\{ f^{(e)} \mathrm{K}_{ij}^{(e)} + \mathrm{M}_{ij}^{(e)} \right\} = sign\left\{ \mathrm{K}_{ij}^{(e)} \right\} \\
& \forall i,\, j \in \{0,1,2\}
\end{aligned} \quad . \tag{5}
$$

- Replace the element diffusion matrix $\mathbf{K}^{(e)}$ by $f^{(e)}\, \mathbf{K}^{(e)}$.

When the dominance of the convection term is not troublesome, the artificial diffusion solution on a relatively coarse initial mesh more or less resembles the exact solution. This is illustrated in figure 3, where the Galerkin solution on a severely refined mesh is compared to the artificial diffusion solution on an initial mesh.

(a)



(b)

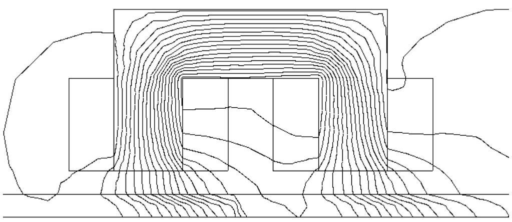


Figure 3: Comparison between the numerical solutions, for a velocity of 20 m/s, (a) by the Galerkin method on a very fine mesh (14423 elements) and (b) by the AD method on a coarse mesh (2046 elements).

5 Adaptive mesh refinement

Artificial diffusion, streamline diffusion and SUPG are used to find a stable approximation for high speed problems on a mesh with elements that are larger than the characteristic size of the transitions. As the artificial or streamline diffusion solution is an approximate solution for another differential equation, it introduces an extra error. This error is reduced by refining the mesh in the transition zones, which implies the use of an error estimator based on the slope of the solution. This slope is calculated as follows:

- Suppose that the equation of the solution in a linear triangular finite element is given by:

$$
A = ax + by + c \quad . \tag{6}
$$

- When the solution in node $(x_i\ ,y_i)$ of the triangle is given by A_i, the coefficients a,b and c are found by:

$$
\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x_0 & y_0 & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} A_0 \\ A_1 \\ A_2 \end{pmatrix} \quad . \tag{7}
$$

- The unit vector $\vec{u}$, perpendicular to the solution in the element is then given by:

$$
\vec{u} = \frac{a\vec{e}_x + b\vec{e}_y - \vec{e}_z}{\sqrt{a^2 + b^2 + 1}} \quad . \tag{8}
$$

- The slope of the solution in the element is high if the z-component of this unit vector is small.

A mesh refinement procedure, based on artificial diffusion, is given in figure 4. The stable first solution is used to detect the regions with the largest decay of the magnetic vector potential in the moving regions. Mesh refinement is only performed for the elements where the slope of the solution is the highest.

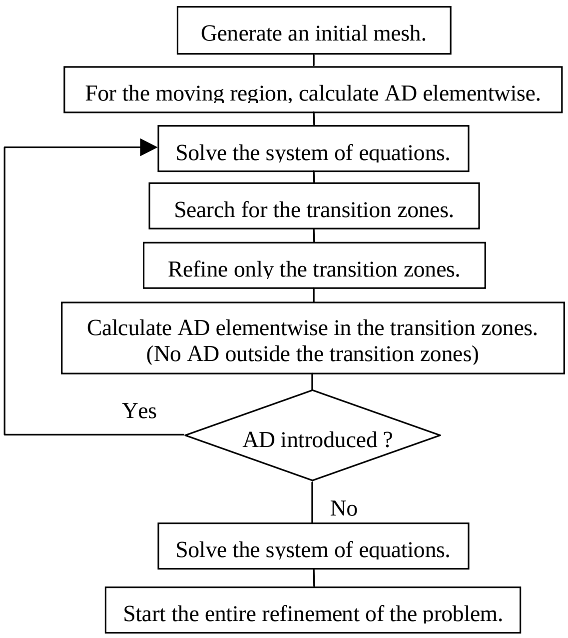


Figure 4: Adaptive mesh refinement procedure for moving regions, based on artificial diffusion.

It can be concluded that the convection term is large in the transition areas, because there the gradient of the solution is high. The convection-diffusion problem thus behaves as a simple diffusion problem outside these transition areas. Therefore, no artificial diffusion is introduced in the following computation steps, except for the transitions areas. To avoid oscillations, the fraction of refined elements in the moving region may not be to low. After a few steps, the elements in the transition areas are so small that artificial diffusion becomes superfluous. Then, the Galerkin method is attained again, and a total refinement can be applied to decrease the global error.

The described procedure allows for finding a locally refined mesh, which does not cause oscillations when the Galerkin method is applied. The same refinement scheme can be used for streamline diffusion. Compared to artificial diffusion, less steps are required because the alteration of the differential equation is smaller.

6 Illustration

In the following numerical example, the rail moves at a speed of 20 m/s to the right side. Figure 5 shows the initial mesh for the artificial diffusion method. In the first step, diffusion is introduced to each element in the moving region. The first approximation with artificial diffusion was already plotted in figure 3b. Comparison of figures 3a and 3b shows that the regions with concentrated field lines are located at different places.

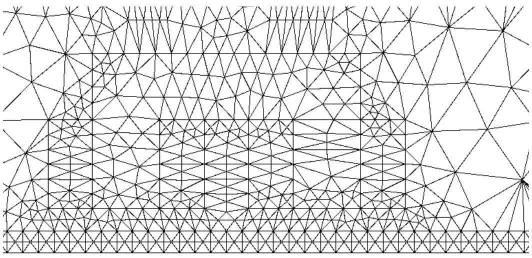


Figure 5: Initial mesh for the artificial diffusion method.

If the fraction of the elements in the moving region that should be refined, is well estimated, mesh refinement is only performed under the iron core. This can be seen in figure 6a. Only in a particular distance from the core, mesh refinement was applied. Figure 6b shows the numerical solution after one refinement step. The solution remains stable, and its steepest regions moved slightly to the border of the rail. Besides a smoothing of the transitions, this indicates that artificial diffusion is moving the transition zones. After three refinement steps, the approximation of figure 7 is calculated. The transition zones moved to the border, and the solution remains stable.

(a)

(b)

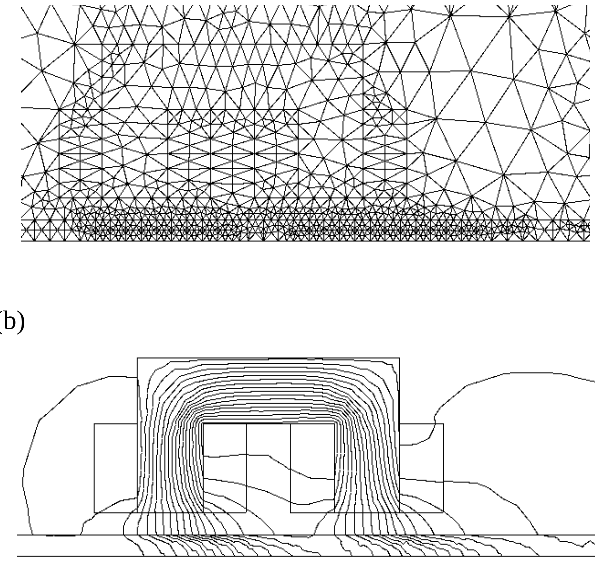


Figure 6: (a) Mesh and (b) artificial diffusion solution, for a velocity of 20 m/s, after the first refinement step.

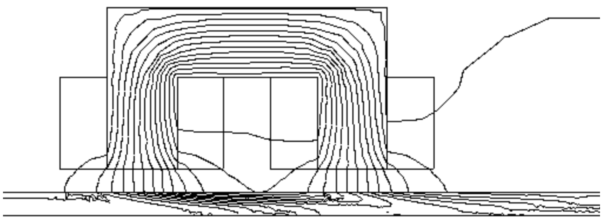


Figure 7: Artificial diffusion solution, for a velocity of 20m/s, after three refinement steps.

7 Possible improvements

As outlined, there are two reasons for improving the refinement procedure:

- Refining a certain percentage of the elements during a few steps yields an extremely huge number of elements.
- During the mesh refinement procedure, the transition zones move.

The extra diffusion (AD or SD) is at its highest level in the initial computation step. After the first adaptation step, a better approximation is obtained, because diffusion is only introduced to the refined areas, where the element size is smaller. Due to the decrease of the extra diffusion the transition areas moved slightly.

To account for this, it is possible to project a new solution on the previous mesh. This mesh is refined again, starting from the new approximation. After a few steps, the difference between two successive solutions vanishes, because the exact positions of the transitions are much better approximated. At this stage, the projection of the solution on a previous mesh is terminated. Then, a successive refinement of the transition areas is performed, to gradually decrease the amount of extra diffusion. Eventually, no extra diffusion is required, and the problem can be solved by the traditional Galerkin method.

8 Conclusion

Three methods to stabilize the finite element method for convection-diffusion problems are presented. Artificial diffusion alters the differential equation, by changing the diffusion constant. Streamline diffusion also alters the differential equation by adding a diffusion term that only acts in the streamline direction. Streamline upwind Petrov Galerkin alters the method, by using other weighting functions. A mesh refinement procedure is described, based on the localization of the regions where the slope of the solution is the highest. This procedure is implemented for artificial diffusion. The same behaviour is expected for streamline diffusion and SUPG. Furthermore, improvements are expected by the projection of intermediate solutions on previous meshes.

9 Acknowledgments

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