Weak Coupling of 2D Magnetical and Mechanical Analysis: Nodal Electromagnetic Forces

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Abstract: A weak coupling between magnetostatic and elasticity equations is derived from energy considerations. The coupling term results directly into a finite element expression for the nodal electromagnetic forces, based upon the partial derivatives of the magnetic stiffness matrix with respect to displacement. The electromagnetic forces are used as source terms for a subsequent elasticity or vibration analysis. The relative contribution of the stator's modal shapes in the deformation excited by this force distribution is calculated. This allows for predicting the machine’s noise and vibrations spectrum at the design level. As an example, the coupling is used to analyse the vibrational behaviour of a 6/4 switched reluctance machine (SRM).

Keywords: finite element methods, electromagnetic forces, modal analysis, mechanical factors, coupled problems.

I. INTRODUCTION

The electromagnetic field inside an electrical machine and its mechanical structure will determine the machine’s behaviour in producing vibrations and noise. The link between the magnetic and the mechanical analysis is the electromagnetical force exerted by the magnetic field on stator and rotor. To predict stator deformations from a given magnetic field, a *local force* formulation is necessary. A finite element based expression for local electromagnetic forces is presented. In the finite element analysis, forces are evaluated at every node of the mesh (nodal forces) and this force distribution is used as an input (source terms) to the subsequent mechanical analysis, static or time-harmonic. From the modal shapes of the stator and the force distribution, *mode participation factors* are determined (as a function of rotor position), indicating the relative importance of the modal shapes towards the machine’s vibrations and noise. This way, the machine’s noise and vibrations spectrum can be anticipated at the design level. This analysis is illustrated in 2D by example of a 6/4 switched reluctance machine (SRM). The method can be readily extended to 3D problems.

II. THE MAGNETO-MECHANICAL SYSTEM

The finite element methods (FEM) for magnetostatic analysis as well as the FEM for elasticity analysis are based upon the minimisation of an energy function.

The *elastic energy* stored in a body with deformation a ($x_i=x_{i,0}+u_i$, $y_i=y_{i,0}+v_i$, $a_i=[u_i\ v_i]^T$) is [1]

$$U = \frac{1}{2} a^T K a \ , \tag{1}$$

where K is the mechanical stiffness matrix, determined by the structure’s geometry and material properties ρ, E and ν, i.e. density, Young modulus and Poisson modulus. The column vector a contains the unknown nodal displacements. The *magnetic energy* stored in an (unsaturated) system with magnetic vector potential A is [2]

$$W = \frac{1}{2} A^T M A = \sum_e \frac{1}{2} A^T M^e A \ , \tag{2}$$

where M is the global magnetic ‘stiffness’ matrix and M^e is the element stiffness matrix, determined by the system’s geometry and magnetic permeability μ. The column vector A contains the unknown magnetic vector potentials. For non-linear (saturated) systems, the magnetic energy differs from the magnetic co-energy and the more general expression

$$W = \sum_e \int_{V^e} \int_0^B H \, dB \, dv \tag{3}$$

has to be used for the energy, where V^e is the element volume, H is magnetic field strength and B is flux density.

Considering the similar form of the energy expressions (1) and (2), it is investigated whether the following combined system of equations can support a coupled magneto-mechanical analysis:

$$\begin{bmatrix} M & D \\ C & K \end{bmatrix} \begin{bmatrix} A \\ a \end{bmatrix} = \begin{bmatrix} T \\ R \end{bmatrix} \tag{4}$$

where T is the magnetical source term vector representing the right hand side of the Poisson equation. R is the mechanical source term vector representing forces other than those of electromagnetic origin (so-called external forces). For clarity, these source terms will be set to zero (T=0, R=0). This means that no external forces are considered, and that the derived force expression is valid only in regions without current density.

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The magnetical and mechanical mesh need not be identical, since projection methods can be used to map the information from one mesh to the other [3].

The coupling matrices C and D can be evaluated considering the total energy E in the magneto-mechanical system (assuming the linear case for now):

$$E = U + W = \frac{1}{2} a^T K a + \frac{1}{2} A^T M A . \quad (5)$$

Setting the partial derivatives of energy E with respect to the unknowns $[A \ \ a]^T$ to zero, gives the combined system (4) with T=0, R=0:

$$\frac{\partial E}{\partial A} = M A + \frac{1}{2} a^T \frac{\partial K(A)}{\partial A} a = 0 , \quad (6)$$

$$\frac{\partial E}{\partial a} = K a + \frac{1}{2} A^T \frac{\partial M(a)}{\partial a} A = 0 . \quad (7)$$

The coupling terms can thus be recognised as

$$D = \frac{1}{2} a^T \frac{\partial K(A)}{\partial A} , \quad (8)$$

$$C = \frac{1}{2} A^T \frac{\partial M(a)}{\partial a} . \quad (9)$$

The coupling term D represents the dependency of mechanical parameters on the magnetic field, e.g. magnetostriction effects. The coupling term C represents the dependency of magnetical parameters on the mechanical displacement, e.g. permeability changes due to density or stress variations. These coupling terms play an important role in the determination of forces related to both effects.

III. Electromagnetic Forces

Using the coupling terms C and D, it is possible to solve the matrix system (4) directly. Solving this strongly coupled system requires an iterative solver that can handle a non-sparse asymmetrical system, e.g. a GMRES solver. Since convergence and computing speed can be expected to be poor for this total matrix, it is useful to examine the weakly coupled version of (4). Moreover, the decoupling will lead to an explicit expression for the electromagnetic nodal forces. For the weakly coupled system, the more common equation solvers for sparse symmetric systems can be used.

When equation (7) is rearranged into

$$K a = -\frac{1}{2} A^T \frac{\partial M(a)}{\partial a} A = -C A , \quad (10)$$

the right hand side is recognised as a force $-CA$ acting on the mechanical subsystem K. Since the external forces were excluded (R=0), this reveals a means to calculate the nodal electromagnetic forces, indicated with F_{em} . The forces F_{em} are calculated from magnetic vector potential A and the partial derivative of the magnetic stiffness matrix M with respect to deformation a:

$$F_{em} = -\frac{1}{2} A^T \frac{\partial M(a)}{\partial a} A = -C A . \quad (11)$$

The coupling term C need not be calculated explicitly to find F_{em}. This expression for F_{em} is also found directly by deriving magnetic energy W (2) with respect to displacement a:

$$F_{em} = -\frac{\partial W}{\partial a} = -\frac{\partial}{\partial a}\left[\frac{1}{2} A^T M A\right] \quad (12)$$

where the unknowns A have to be considered constant (virtual work principle) but where M is a function of a. For the non-linear case, the nodal force expression becomes, combining (3) and (12) [4]:

$$F_{em} = -\frac{\partial}{\partial a} \sum_e \iint_{V^e} \int_0^B H \, dB \, d\mathrm{v} \quad (13)$$

$$= -\sum_e \left\{ \int_{V^e} \left(H \frac{\partial B}{\partial a} \right) d\mathrm{v} + \int_{V^e} \left(\int_0^B H \, dB \right) \frac{\partial \mathrm{v}}{\partial a} \right\} . \quad (14)$$

IV. Magnetostriction

Similar considerations can be made concerning the coupling term D to evaluate forces related to magnetostriction. Rearranging (6) into

$$M A = -\frac{1}{2} a^T \frac{\partial K(A)}{\partial A} a = -D a , \quad (15)$$

allows for interpreting the right hand side as a current density source $-Da$ for the magnetical subsystem M. Since 'external' current density sources were excluded (T=0), this reveals a means to represent magnetostrictive effects by equivalent current densities, indicated with J_m . The current sources J_m are calculated from mechanical displacement a and the partial derivative of the mechanical stiffness matrix K with respect to magnetic vector potential A:

$$J_m = -\frac{1}{2} a^T \frac{\partial K(A)}{\partial A} a = -D a . \quad (16)$$

The coupling term D need not be calculated explicitly to find J_m. This expression for J_m is also found directly by deriving elastic energy U (1) with respect to magnetic vector potential A:

$$J_m = -\frac{\partial U}{\partial A} = -\frac{\partial}{\partial A}\left[\frac{1}{2}a^T K a\right] \tag{17}$$

where the unknowns a have to be considered constant (virtual work principle) but where K is a function of A. Note that the current densities J_m are the nodal representations of the element's current density $J^{(e)}$ (with Δ the element area):

$$J_m = \frac{J^{(e)}\Delta}{3}. \tag{18}$$

If it is assumed that the mechanical material properties E, ν and ρ do not depend on vector potential A (neglecting magnetostriction), then the coupling term D vanishes. When additionally the coupling term C is replaced by the force F_{em} on the right hand side, the system (4) is decoupled into

$$\begin{bmatrix} M & 0 \\ 0 & K \end{bmatrix}\begin{bmatrix} A \\ a \end{bmatrix} = \begin{bmatrix} T \\ R + F_{em} \end{bmatrix}. \tag{19}$$

Equation (19) can be solved using a simple cascade procedure and its validity has been succesfully tested against analytical models.

V. THE PARTIAL DERIVATIVE $\partial M/\partial a$

The derivation $\partial M/\partial a$ is illustrated briefly for the case of first order 2D triangular elements. For the magnetic element matrix [5]

$$M_{ij}^e = \frac{1}{4\mu\Delta}\left[b_i b_j + c_i c_j\right] \tag{20}$$

with permeability μ, element area Δ and the familiar shape function coefficients $a_1 = x_2y_3 - x_3y_2$, $b_1 = y_2 - y_3$, $c_1 = x_3 - x_2$, the partial derivative with respect to u_1 is

$$\frac{\partial M_{ij}^e}{\partial u_1} = \frac{1}{4\mu\Delta}\begin{bmatrix} 0 & c_1 & -c_1 \\ c_1 & 2c_2 & c_3 - c_2 \\ -c_1 & c_3 - c_2 & -2c_3 \end{bmatrix} - \frac{2b_1}{\Delta} M_{ij}^e. \tag{21}$$

This derivative contains two terms since not only the coefficients a_i, b_i and c_i but also the element area Δ depend on the nodal displacements. The magnetic permeability μ was assumed constant here (independent of stress); in the other case, the chain rule will add a third term. Similar expressions are found for the partial derivative of M_{ij} with respect to u_2, u_3, v_1, v_2 and v_3. These six derivatives give the contribution of this element to the x and y components of the force on the three element nodes.

Rather than calculating a finite energy difference between two finite element solutions, the partial derivative represents more accurately the essence of virtual work [6]. There is no need for a second magnetic finite element solution and no numerical derivations are performed.

VI. EXAMPLE: 6/4 SRM

A. Nodal Forces

The geometry of the example 6/4 SRM is shown in Fig.1a for one specific rotor position. The coil system indicated is current excited and generates the magnetic field shown in Fig.1b. This magnetic field is used to evaluate the local electromagnetic forces F_{em} using (11). The forces on the stator structure are given in Fig.2. For this rotor position, the machine produces torque due to the (net) forces alongside the stator teeth. The radial forces are larger than the tangential forces. The radial forces do not produce torque but do cause stator deformation.

B. Mode Shapes and Modal Participation

Using the stator's mechanical matrices K (stiffness), M_m (mass) and C_m (damping), the eigenvectors and eigenvalues of the mechanical structure are found. These constitute the stator deformation mode shapes. Several 2D mode shapes are shown in Fig.3. A more accurate analysis will have to involve 3D mode shapes [7]. Table 1 lists the first 20 modes with their frequencies and their (normalised) participation factor for the force distribution under consideration. The mode participation factor is determined using [8]

$$\Gamma_i = \frac{1}{m_i \phi_i^2} \sum_j \phi_i(x_j)^T p(x_j), \tag{22}$$

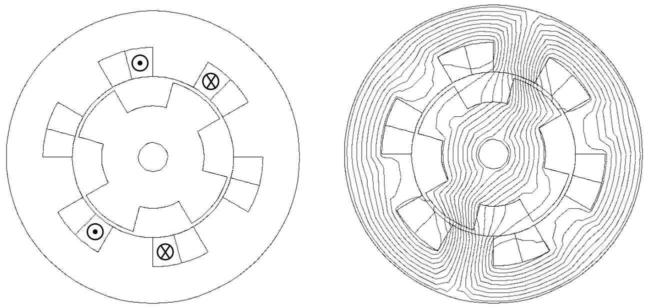

Fig. 1. a) Geometry of the 6/4 SRM and b) equipotential lines for excitation according to a).

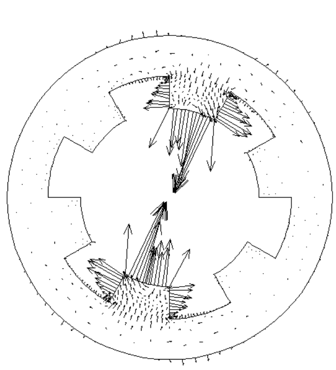

Fig. 2. Nodal force distribution calculated from the magnetic field.

where $\phi_i(x_j)$ is the displacement of the i^{th} mode shape at the j^{th} node, $p(x_j)$ is the force at the j^{th} node and m_i is the generalised mode mass.

In Table 1, several modes are paired because they represent deformation patterns that differ an angular shift only, e.g. modes (1,2) are shifted over 90°, modes (4,5) and modes (9,10) over 45°. The machine's mounting is not considered in determining the participation factors.

From Table 1 it is seen that the 8th mode shape, the uniform shrinking and expanding of the stator structure, has the largest contribution and determines the greater part of the vibrational behaviour of the machine. The squaring modes (9,10) have a substantial contribution and also the ovalization modes (4,5) are clearly present. The triangular modes 6 and 7 have no significant contribution since there is no triangular symmetry in the forces. The contribution of the 3rd mode shape, the rigid body rotation, is a measure for the instantaneous torque of the SRM. When this value is calculated for all rotor positions, it can be used as an indication for the machine's torque and torque ripple.

TABLE 1. PARTICIPATION FACTORS OF MODAL SHAPES IN FORCE DISTRIBUTION.

MODE NUMBER	FREQUENCY (HZ)	MODE PARTICIPATION FACTOR (NORMALISED)
1, 2	rigid translations	0.0270
3	rigid rotation	0.0227
4, 5	334.9	0.4676
6	783.8	0.0269
7	1028.1	0.0005
8	1311.3	0.5810
9, 10	1598.5	0.4941
11, 12	1862.3	0.1286
13, 14	2488.7	0.0671
15	2851.3	0.0211
16, 17	3249.2	0.3114
18	4019.5	0.1209
19, 20	4530.6	0.2542

The force distribution in Fig.2 is only a snapshot for this specific rotor position: a full modal analysis has to consider the force distribution for several rotor positions, corresponding to different time steps and coil excitations. When the participation factors Γ_i are known for different rotor positions (and time instants), the set of equations

$$\ddot{q}_i + 2\zeta_i \omega_i \dot{q}_i + \omega_i^2 q_i = \Gamma_i(t) \qquad (23)$$

can be solved for a selected set of significant modes ϕ_i, giving their generalised co-ordinates q_i as a function of time (ω_i = eigenfrequency, ζ_i = modal damping factor). Only when the mechanical damping C_m is assumed to be proportional, the system of equations (23) can be decoupled and solved separately [8].

VII. CONCLUSIONS

A weak coupling between magnetical and mechanical analysis is derived, leading to a finite element based expression for the nodal electromagnetic forces. This expression is based on partial derivatives of the magnetic stiffness matrix (true virtual work) and leads to a distribution of local forces. This force pattern can be used for elastic or vibrational analysis or any other post-processing action. The stator modal shapes and their participation factor in the force distribution are calculated for one rotor positions and a single time instant. Repeating this analysis for all rotor positions allows for anticipating the stator resonances and the noise frequency spectrum at the design level.

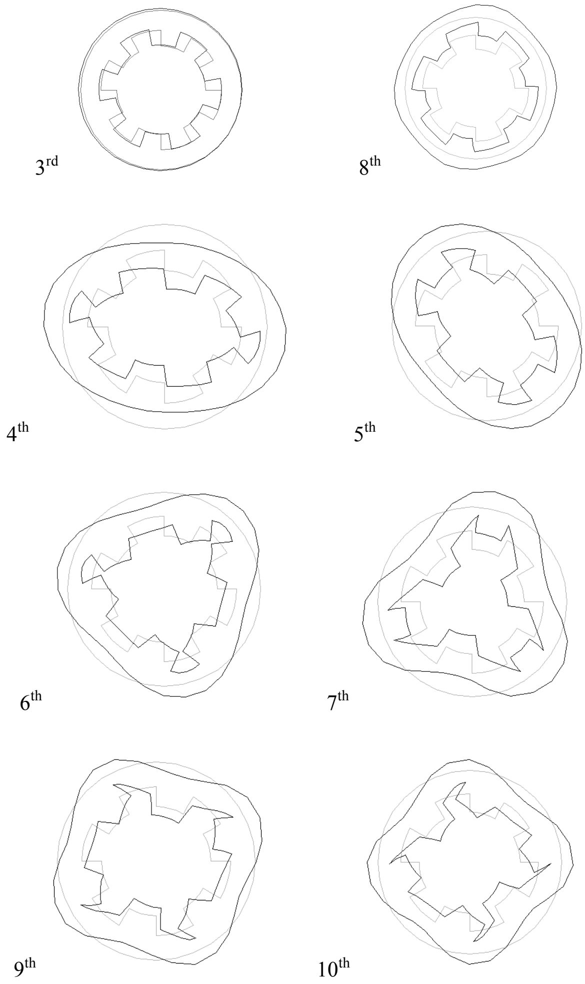

Fig. 3. Selected modal shapes for the 6/4 SMR stator structure. The mode numbers are assigned according to ascending frequency.

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