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A Topological Method used for Field-Circuit Coupling

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***Abstract* — A lumped parameter model describing the outside electric connection of conductors is linked with a finite element model to compute the magnetic field distribution. A well-established method for circuit analysis is adapted for this purpose. A suitable analysis to consider solid conductors and stranded conductors simultaneously as parts of the external electric circuit is stated. This approach yields a general coupling mechanism that keeps symmetry if both the magnetic and the electric problems are symmetric. The generality of the method makes the implementation straightforward and powerful. The maturity of the method is proved by the computation of different eddy current problems.**

***Index terms* — eddy currents, finite element methods, signal flow graphs.**

I. Introduction

The magnetic field of an eddy current problem depends on the way of connecting the solid conductors and stranded conductors of the magnetic finite element (FE) model with external impedances, voltage sources and current sources. As the main interest goes to the magnetic field distribution, the circuit connection is often seen as an extra boundary condition [1]. Extra equations are added for current driven solid conductors and voltage driven stranded conductors [2]-[5]. This approach treats circuit connections fragmentarily. In most technical cases, however, the conductors are joined together in a complex way [1]. The load of the physical system enters the model via the electric circuit. The stability of the solution is strongly related to the properties of the electric circuit [1][4][6]. These remarks affirm the importance of a general and robust method for modelling and coupling the external circuits.

II. Magnetic Model

The magnetic flux density $\mathbf{B}$ and the electric field strength $\mathbf{E}$ are written in terms of the magnetic vector potential $\mathbf{A}$ and the electric scalar potential V [7]. Ampère's and Faraday's laws give the partial differential equation

$$\nabla \times \nu \nabla \times \mathbf{A} + \sigma \frac{\partial \mathbf{A}}{\partial t} = -\sigma \nabla V \qquad (1)$$

where ν is the magnetic reluctivity and σ is the electric conductivity. The Galerkin form of (1) is discretized by means of finite elements for a 2D quasi-static model (2) and a 2D problem considering motional eddy current effects (3).

$$\sum_j \left(K_{ij} + \underline{j}\omega L_{ij} \right) A_j - \sum_{p=1}^{n_{str}} P_{i,p} I_{str,p} - \sum_{q=1}^{n_{sol}} Q_{i,q} V_{sol,q} = 0 \qquad (2)$$

$$\sum_j \left(K_{ij} + M_{ij} \right) A_j - \sum_{p=1}^{n_{str}} P_{i,p} I_{str,p} - \sum_{q=1}^{n_{sol}} Q_{i,q} V_{sol,q} = 0 \qquad (3)$$

where

$$K_{ij} = \int_\Omega \nu \nabla N_i \nabla N_j \, \mathrm{d}\Omega \, , \; L_{ij} = \int_\Omega \sigma N_i N_j \, \mathrm{d}\Omega \, ,$$

$$M_{ij} = \int_\Omega \sigma \mathbf{v}.\nabla N_j N_i \, \mathrm{d}\Omega \, , \; P_{i,p} = \int_{\Omega_{str,i}} \frac{N_{t,i}}{\Delta_{str,i}} N_i \, \mathrm{d}\Omega$$

$$\text{and } Q_{i,q} = \int_{\Omega_{str,q}} \frac{\sigma}{\ell} N_i \, \mathrm{d}\Omega \, . \qquad (4)$$

N_i is the form function associated with the mesh node i. n_{str} and n_{sol} are the numbers of stranded and solid conductors respectively. $I_{str,p}$ is the current per strand, $N_{t,p}$ is the number of turns and $\Delta_{str,p}$ is the cross-section of stranded conductor p. $V_{sol,q}$ is the voltage drop and $\Delta_{sol,q}$ is the cross-section of solid conductor q. $\mathbf{v}$ is the speed. ℓ is the active length of the magnetic model. In stranded conductors, no skin effect appears. Therefore, the contributions of the cross-sections of the stranded conductors to L_{ij} and M_{ij} are zero [3].

III. Circuit Conditions

As long as all currents inside the stranded conductors and all solid conductor voltage drops are known, no extra unknowns have to be added to the system of equations. Voltage driven stranded conductors and current driven solid conductors cause $I_{str,p}$ respectively $V_{sol,q}$ to be unknowns. The voltage drop over a stranded conductor p is

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$$V_{str,p} = \underbrace{\frac{N_{t,p}{}^{2}\ell}{\sigma f_{str,p}\Delta_{str,p}}}_{R_{str,p}} I_{str,p} + \underline{j}\omega\ell \sum_{j} P_{j,p} A_j \tag{5}$$

where $f_{str,p}$ is the fill factor of the stranded conductor [8]. The total current through a solid conductor q is

$$I_{sol,q} = \underbrace{\frac{\sigma\Delta_{sol,q}}{\ell}}_{G_{sol,q}} V_{sol,q} - \underline{j}\omega\ell \sum_{j} Q_{j,q} A_j \ \ [8]. \tag{6}$$

If all stranded conductors are voltage driven and if all solid conductors are current driven, the coupled matrix is

$$\left[\begin{array}{c|cc} K_{ij} & -P_{i,p} & -Q_{i,q} \\ \hline \underline{j}\omega\ell P_{j,p} & R_{str,p} & 0 \\ -\underline{j}\omega\ell Q_{j,q} & 0 & G_{sol,q} \end{array}\right] \cdot \left[\begin{array}{c} A_j \\ \hline I_{str,p} \\ V_{sol,q} \end{array}\right] = \left[\begin{array}{c} 0 \\ \hline V_{str,p} \\ I_{sol,q} \end{array}\right]. \tag{7}$$

Symmetry is obtained by multiplying (6) with $\chi = {}^{1}\!/_{\underline{j}\omega\ell}$ and (5) with $-\chi$ [3,9].

IV. External Circuits

In reality, magnetic branches are connected to each other respectively to several sources and impedances. Circuit analysis translates the circuit in a systems of equations. The coupling to a magnetic FE description requires that $I_{str,p}$ and $V_{sol,q}$ are circuit unknows or can be derived from them.

Consider the circuit in Fig. 1. A nodal circuit analysis associates a voltage V_m to each node m. For each connected part of the circuit one reference node $V_{m0} = 0$ is chosen. This approach fulfils the Kirchoff Voltage Law (KVL) explicitly. For each node the Kirchoff Current Law (KCL) is written. In the Modified Nodal Analysis (MNA), this is immediately performed in terms of the nodal voltages and the branch admittances. A problem occurs for voltage sources and stranded conductors. Here, an extra unknown current is added to the system. An extra equation expresses the voltage drop.

$$\left[\begin{array}{cc|c} \cdot & \cdot & 1 \\ \cdot & \cdot & -1 \\ \hline 1 & -1 & 0 \end{array}\right] \cdot \left[\begin{array}{c} V_m \\ V_n \\ \hline i_{mn} \end{array}\right] = \left[\begin{array}{c} \cdot \\ \cdot \\ \hline V_v \end{array}\right] \tag{8}$$

This method generates a zero-diagonal element that may make the solving of the system of equations troublesome.

The Compact MNA eliminates both, the unknown current and one of the nodal voltages by an appropriate substitution. The resulting system is symmetric. The substitution of the dense circuit conditions for stranded conductors in the sparse FE equations, results in a significant loss of matrix sparsity.

A compacted branch analysis generates the same problem but for solid conductors. It is obvious that the optimal approach of the circuit coupling is a description in terms of both, unknown currents and unknown voltages.

V. Signal Flow Graph

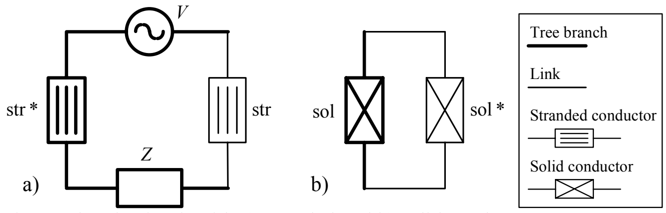

Fig. 1. Electric circuit with a) stranded and b) solid conductors.

A topological method for circuit analysis, like the Signal Flow Graph (SFG) method, derives a graph structure out of the circuit. Operations on the graph reduce the problem and enable an optimal implementation of this method. A SFG is a weighted and directed graph representing a system of equations [10]. The value of a node equals the sum of all node values from which the incoming arrows originate, weighted with the arrow weights. A *source node* is a node with only outgoing arrows. A *dependent node* is a node with at least one incoming arrow. A *sink node* is a node with only incoming arrows. The SFG represents a symmetric system if the paths between two dependent nodes in both directions have the same weights. The extraction of the matrix equations out of a SFG is straightforward. The unknown graph nodes become system unknowns. The dependent graph nodes represent matrix equations.

A *loop* is a circuit path with the same begin and end node. A *cutset* is a set of branches which removal splits the circuit in two parts. A *tree* is a set of branches that connects all nodes and contains no loops. The remaining branches are forming the *cotree*. A *link* is a cotree branch. A *fundamental loop* is a loop formed by one link and a set of tree branches. A *fundamental cutset* is a cutset formed by one tree branch and a set of links [11]. It is assumed that the set of the voltage sources does not contain loops and that the set of the current sources does not contain cutsets.

A tree is traced through the circuit with respect to the privileged order: voltage sources, solid conductors, admittances and stranded conductors. As a consequence, the privileged order of cotree members is: current sources, stranded conductors, impedances and solid conductors. In Fig. 1 the bold lines represent tree branches, whereas the thin lines represent links.

The SFG is built by putting a node for each voltage and current (Fig. 2) [12]. The KCL and KVL are expressed by arrows between the nodes. Joining graphs does not change the graph nodes as long as the dependent nodes of the first graph correspond to source nodes of the other graph and vice versa. Therefore, branch current-voltage relations (BCVR) are added either as impedances or admittances (Fig. 2) [12]. Two dependent nodes are joined together to a zero node by changing the sign of all incoming branch weights of one of the subgraphs. This happens for the stranded conductor links and the solid conductor tree branches (Fig. 3).

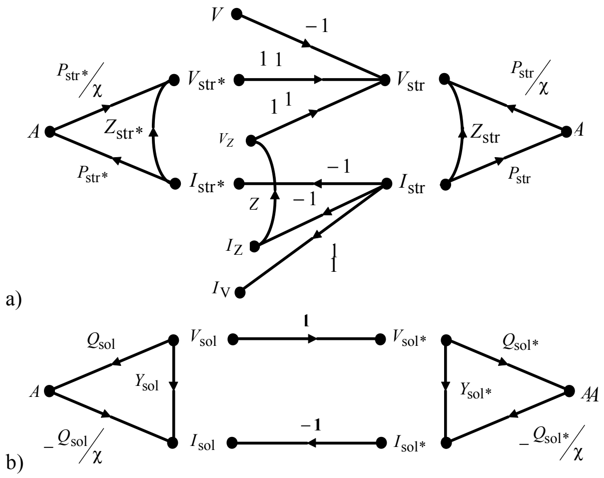

Fig. 2. Non-coupled Signal Flow Graph.

VI. Matrix Formulation

The *fundamental cutset matrix* **D** represents the incidences of the circuit branches to the fundamental cutsets. The *fundamental loop matrix* **B** represents the incidences of the circuit branches to the fundamental loops [11]. Each of the matrices is partitioned in parts associated with the stranded conductors that are links (*str*), those that are tree branches (*str**), the solid conductors that are tree branches (*sol*), those that are links (*sol**), the independent sources (*i* and *v*) and the immittance tree branches (*T*) and links (*L*). Table I shows the equivalences between the SFG and the matrix calculus.

TABLE I
Equivalence Between Circuit Theory, Signal Flow Graph and Matrix Calculus

Circuit	Signal Flow Graph	Matrix notation
KCL	current nodes	$\mathbf{DI} = 0$
KVL	voltage nodes	$\mathbf{BV} = 0$
BCVR	vertical connections	$\mathbf{V} = \mathbf{ZI}$; $\mathbf{I} = \mathbf{YV}$
α	eliminate $\mathbf{V}_{str*}$, $\mathbf{I}_{str*}$	$\mathbf{I}_{str*} = -\mathbf{D}_{str*,i}\mathbf{I}_{\mathrm{i}} - \mathbf{D}_{str*,str}\mathbf{I}_{str}$
β	eliminate $\mathbf{V}_{sol*}$ $\mathbf{I}_{sol*}$	$\mathbf{V}_{sol*} = -\mathbf{B}_{sol*,v}\mathbf{V}_{v} - \mathbf{B}_{sol*,sol}\mathbf{V}_{sol}$
γ	eliminate $\mathbf{V}_{str}$, $\mathbf{V}_{L}$, $\mathbf{I}_{sol}$, $\mathbf{I}_{T}$	$\mathbf{V}_{L} = \mathbf{Z}_{L}\mathbf{I}_{L}$; $\mathbf{I}_{T} = \mathbf{Y}_{T}\mathbf{V}_{T}$

VII. Graph Contraction

The mentioned way of combining magnetically coupled branches with the electric circuit causes three difficulties:

- stranded conductor tree branches are described by a dependent current,
- solid conductor links are described by a dependent voltage and
- the coupling terms are not symmetric.

Three operations solve these problems:

1. *Partial cutset transformation*

The precedences while choosing tree branches cause that a fundamental cutset associated with a stranded conductor tree branch only contains current sources and stranded conductors. A partial cutset transformation contracts the graph in direction α (Fig. 3). The current of the stranded conductor tree branch is expressed as a combination of independent currents and other stranded conductor currents (Table I).

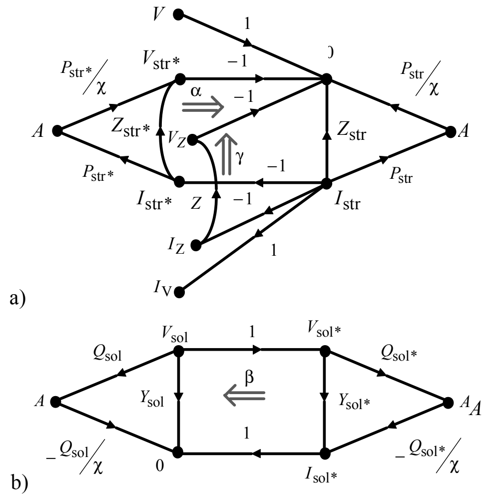

Fig. 3. Coupled Signal Flow Graph.

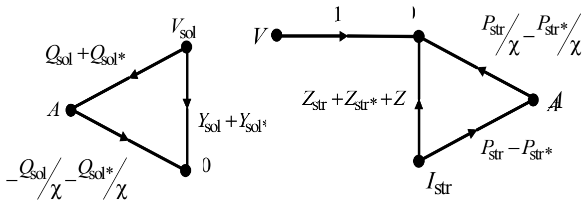

Fig. 4. Compact Signal Flow Graph.

1. *Partial loop transformation*

In an analogue way, a fundamental loop associated with a solid conductor link only exists of voltage sources and solid conductors. A partial loop transformation contracts the graph in direction β (Fig. 3).

1. *Symmetrizing the system*

A contraction of the BCVR (direction γ in Fig. 3) leads to a Compact Signal Flow Graph (Fig. 4).

Multiplying the circuit loop equations with χ and the circuit cutset equations with −χ leads to the coupled field-circuit matrix

$$\begin{bmatrix} \mathbf{K} & -\mathbf{F} \\ -\mathbf{F}^{\mathrm{T}} & \chi\mathbf{S} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{A} \\ \mathbf{C} \end{bmatrix} = \begin{bmatrix} \mathbf{H} \\ \chi\mathbf{W} \end{bmatrix} \tag{9}$$

where $\boldsymbol{K}$ is $\left[K_{ij} + j\omega L_{ij}\right]$ for a time-harmonic field or $\left[K_{ij} + M_{ij}\right]$ to consider motion, $\boldsymbol{A}$ is $\left[A_j\right]$,

$$\mathbf{S} = \begin{bmatrix} -\mathbf{R}^{*}_{str} & \mathbf{0} & -\mathbf{B}_{str,sol} & -\mathbf{B}_{str,T} \\ \mathbf{0} & -\mathbf{Z}_{L} & -\mathbf{B}_{L,sol} & -\mathbf{B}_{L,T} \\ \mathbf{D}_{sol,str} & \mathbf{D}_{sol,L} & \mathbf{G}^{*}_{sol} & \mathbf{0} \\ \mathbf{D}_{T,str} & \mathbf{D}_{T,L} & \mathbf{0} & \mathbf{Y}_{T} \end{bmatrix} \tag{9a}$$

with

$$\mathbf{R}^*_{str} = \mathbf{R}_{str} - \mathbf{B}_{str,str*}\mathbf{R}_{str*}\mathbf{D}_{str*,str} , \tag{9b}$$

$$\mathbf{G}^*_{sol} = \mathbf{G}_{sol} - \mathbf{D}_{sol,sol*}\mathbf{G}_{sol*}\mathbf{B}_{sol*,sol} \text{ and} \tag{9c}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{I}_{str} & \mathbf{I}_L & \mathbf{V}_{sol} & \mathbf{V}_T \end{bmatrix}^{\mathrm{T}} , \tag{9d}$$

$$\mathbf{H} = -\mathbf{P}_{str*}\mathbf{D}_{str*,i}\mathbf{I}_i - \mathbf{Q}_{sol*}\mathbf{B}_{sol*,v}\mathbf{V}_v , \tag{9e}$$

$$\mathbf{F} = \begin{bmatrix} \mathbf{P}_{str} - \mathbf{P}_{str*}\mathbf{D}_{str*,str} & \mathbf{0} & \mathbf{Q}_{sol} - \mathbf{Q}_{sol*}\mathbf{B}_{sol*,sol} & \mathbf{0} \end{bmatrix} \tag{9f}$$

and

$$\mathbf{W} = \begin{bmatrix} \mathbf{B}_{str,v}\mathbf{V}_v - \mathbf{B}_{str,str*}\mathbf{R}_{str*}\mathbf{D}_{str*,i}\mathbf{I}_i \\ \mathbf{B}_{L,v}\mathbf{V}_v \\ -\mathbf{D}_{sol,i}\mathbf{I}_i + \mathbf{D}_{sol,sol*}\mathbf{G}_{sol*}\mathbf{B}_{sol*,v}\mathbf{V}_v \\ -\mathbf{D}_{T,i}\mathbf{I}_i \end{bmatrix} . \tag{9g}$$

In the case of a quasi-static problem, $\mathbf{K}$ is complex symmetric [9,7]. $\mathbf{S}$ is symmetric because of the property $\mathbf{B}_{x,y} = -\mathbf{D}^{\mathrm{T}}_{y,x}$ [11] and due to the fact that $\mathbf{R}_{str}$, $\mathbf{R}_{str*}$, $\mathbf{G}_{sol}$, $\mathbf{G}_{sol*}$, $\mathbf{Z}_L$ and $\mathbf{Y}_T$ are diagonals.

VIII. NUMERICAL EXAMPLES

Two eddy current problems are presented. A conducting plane is moving between two inductors (Fig. 5). The conductors are connected as in Fig. 1. In the second example an induction machine operated under load (Fig. 6) is connected to the circuit of Fig. 7 [6]. The numbers of additional circuit equations of the different methods are collected in Table II.

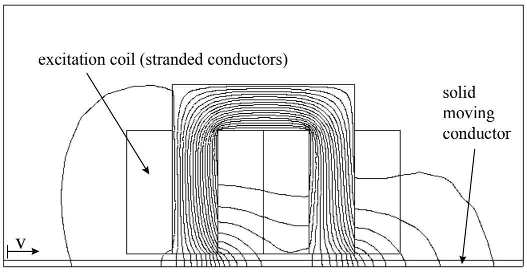

Fig. 5. Equipotential plot of an inductor and a conducting plane moving at 10 m/s to the right.

TABLE II
NUMBER OF CIRCUIT EQUATIONS

Circuit Analysis	Tableau	MNA	CMNA	SFG
Inductor	13	5	4	2
Induction motor	166	46	43	31

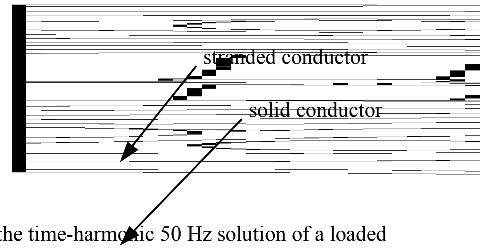

Fig. 6. Flux line plot of the time-harmonic 50 Hz solution of a loaded induction motor.

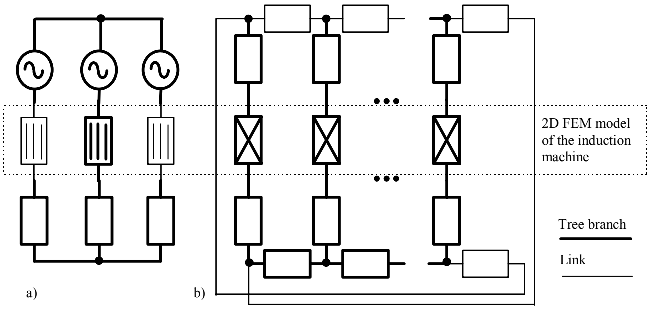

Fig. 7. Electric circuit of a) the stator and b) of the rotor of an induction motor.

IX. CONCLUSION

The Signal Flow Graph method offers a general and robust way to couple a lumped parameter model to a finite element model. The method adds a minimal amount of extra equations to the system matrix. Both, the stranded and solid conductors, arbitrarily connected in the network, fit in the description. Matrix symmetry is retained. This graph method is easily extended to non-linear circuits and circuits with dependent sources. An appropriate modellization of such circuits coupled with a transient FEM is in investigation.

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