Force Calculation Based on a Local Solution of Laplace's Equation

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Abstract - **The finite element method in two dimensions is a common technique to analyse electromagnetic devices. The Maxwell stress method to calculate forces has an important role in this analysis. To obtain an acceptable accuracy of local field values, higher order elements are required yielding an increased computation time. A local solution of Laplace's equation with the finite element solution as boundary conditions, promises a higher accuracy than the conventional method. The combination with lower order elements gives a good trade-off between accuracy and computation time. Two different methods are compared in the analysis of two applications.**

I. Introduction

Computing forces acting on a contour C with the Maxwell stress tensor, requires local values of the normal and tangential components of the magnetic flux density $\vec{B}$.

$$F_n = \frac{1}{2\mu_0} \int_C \left(B_n^2 - B_t^2 \right) \mathrm{d}\, s \tag{1a}$$

$$F_t = \frac{1}{\mu_0} \int_C B_n B_t \, \mathrm{d}\, s \tag{1b}$$

The two-dimensional finite element solution approximates the z-component of the magnetic vector potential $\vec{A}$ as a polynomial of order m over each element. Due to numerical differentiation, the order of the flux density is of one less, i.e. (m-1). This causes numerical inaccuracy of the conventional Maxwell stress method when first order elements are used.

A local solution of Laplace's equation in circular co-ordinates defined by one or two circular boundaries, results in a higher accuracy of local field values [2, 4].

$$r^2 \frac{\partial^2 A_z(r,\rho)}{\partial r^2} + r \frac{\partial A_z(r,\rho)}{\partial r} + \frac{\partial^2 A_z(r,\rho)}{\partial \rho^2} = 0 \tag{2}$$

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The general solution of Laplace's equation in circular co-ordinates is given as a series of circular harmonics of order k [1]:

$$A_z(r,\rho) = \frac{a_0}{2} + \sum_{k=1}^{N} \Big(a_k\, r^k \cos(k\,\rho) + b_k\, r^k \sin(k\,\rho) + c_k\, r^{-k} \cos(k\,\rho) + d_k\, r^{-k} \sin(k\,\rho) \Big) \tag{3}$$

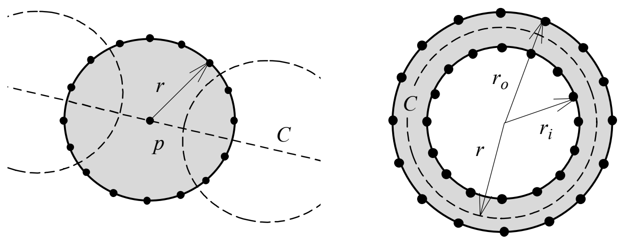
II. Methods

A. First Method

To determine the coefficients in (3), two different methods can be used. The first method [2, 4] uses the known values of the magnetic vector potential on a circle with radius r as boundary conditions. The local field value at the centre point p of the circular source free region is calculated. This centre point is part of a contour C of arbitrary shape (Fig. 1a). The different points on the contour are chosen in such a way that the circles overlap. Because of a finite-value of the magnetic vector potential in the centre point p ($r = 0$), the coefficients c_k and d_k are zero. A value for the magnetic flux density at the centre point is necessary to calculate the force. Due to the arbitrary shape of the contour, the magnetic flux density is decomposed in its x- and y-components.

$$B_x = \left. \frac{\partial A(x,y)}{\partial y} \right|_p = b_1 \tag{4a}$$

$$B_y = \left. -\frac{\partial A(x,y)}{\partial x} \right|_p = -a_1 \tag{4b}$$



a) 1st method b) 2nd method

Fig. 1. Source free region for both methods.

B. Second Method

The second method uses the values of the magnetic vector potential on two concentric circles with radii r_i and r_o as boundary conditions (Fig. 1b). Local field values on the circular contour C with radius $r_i < r < r_o$ are calculated.

If the inner radius r_i is taken as reference and a_0 is assumed to be zero, the general solution of Laplace's equation is

$$A_z(r,\rho) = \sum_{k=1}^{N}\left(a_k\left(\frac{r}{r_i}\right)^k \cos(k\rho) + b_k\left(\frac{r}{r_i}\right)^k \sin(k\rho) + c_k\left(\frac{r_i}{r}\right)^k \cos(k\rho) + d_k\left(\frac{r_i}{r}\right)^k \sin(k\rho)\right). \quad (5)$$

The coefficients a_k, b_k, c_k and d_k are independently determined for each circular harmonic. A FFT algorithm is used to express the magnetic vector potential at the boundaries as a series of such circular harmonics.

$$A_z(r_i,\rho) = \sum_{k=1}^{N}\left(a_{k,i}\cos(k\rho) + b_{k,i}\sin(k\rho)\right) \quad (6a)$$

$$A_z(r_o,\rho) = \sum_{k=1}^{N}\left(a_{k,o}\cos(k\rho) + b_{k,o}\sin(k\rho)\right) \quad (6b)$$

$$\begin{bmatrix} 1 & 1 \\ \left(\frac{r_o}{r_i}\right)^k & \left(\frac{r_i}{r_o}\right)^k \end{bmatrix}\begin{bmatrix} a_k \\ c_k \end{bmatrix} = \begin{bmatrix} a_{k,i} \\ a_{k,o} \end{bmatrix} \quad (7a)$$

$$\begin{bmatrix} 1 & 1 \\ \left(\frac{r_o}{r_i}\right)^k & \left(\frac{r_i}{r_o}\right)^k \end{bmatrix}\begin{bmatrix} b_k \\ d_k \end{bmatrix} = \begin{bmatrix} b_{k,i} \\ b_{k,o} \end{bmatrix} \quad (7b)$$

Once the magnetic vector potential is known at the contour C, the normal and tangential component of the magnetic flux density can be determined.

$$B_n(r,\rho) = \sum_{k=1}^{N}\left(-k\, a_k \frac{r^{k-1}}{r_i^k}\sin(k\rho) + k\, b_k \frac{r^{k-1}}{r_i^k}\cos(k\rho) - k\, c_k \frac{r_i^k}{r^{k+1}}\sin(k\rho) + k\, d_k \frac{r_i^k}{r^{k+1}}\cos(k\rho)\right) \quad (8a)$$

$$B_t(r,\rho) = \sum_{k=1}^{N}\left(-k\, a_k \frac{r^{k-1}}{r_i^k}\cos(k\rho) - k\, b_k \frac{r^{k-1}}{r_i^k}\sin(k\rho) + k\, c_k \frac{r_i^k}{r^{k+1}}\cos(k\rho) + k\, d_k \frac{r_i^k}{r^{k+1}}\sin(k\rho)\right) \quad (8b)$$

The tangential force component F_t results in the torque T of the device. It can be shown that the value of the torque is given by (9) and is independent of de radius r of the contour C. It is not necessary to calculate the normal and tangential component of the magnetic flux density on the contour resulting in a faster algorithm.

$$T = \frac{2\pi}{\mu_0}\sum_{k=1}^{N}\left(k^2\left(b_k\, c_k - a_k\, d_k\right)\right) \quad (9)$$

III. APPLICATIONS

The first method allows arbitrary shaped contours. Forces can be calculated on straight and circular contours. The second method has an advantage for the force calculation in rotating machines with a small air gap. The first method requires in this case a large number of small overlapping circles, increasing the computation time.

Fig. 2 shows the result of a finite element calculation of a circular permanent magnet centred in an iron yoke. The magnetization direction is 45°. Fig. 3 shows the result of the numerical calculation of the x-component of the magnetic flux density for one circular region. Because the magnetic flux density is constant over an element (first order shape functions), the elements are visible. Fig. 4 shows the computed results of the Laplace based approximation. The surface is much smoother resulting in more accurate values of force and torque. Zero force and torque are obtained for a rougher mesh compared to the numerical differentiation.

A typical example of a rotating machine with a small air gap is a squirel cage induction machine. To deal with the non-linearity of the iron and the induced currents in the rotor, a non-linear time-harmonic solution is required. Fig. 5 shows the finite element model of one quarter of a 4-pole induction machine with 36 stator slots and 28 rotor bars. The air gap consists of 3 layers of elements.

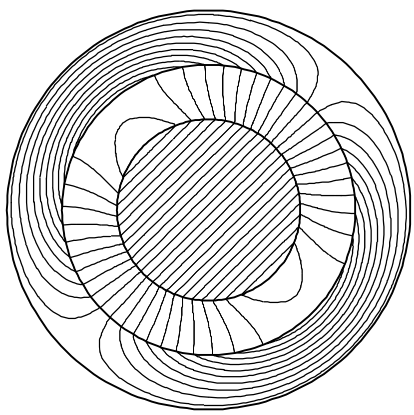


Fig. 2. Equipotential lines of a permanent magnet centred in an iron yoke.

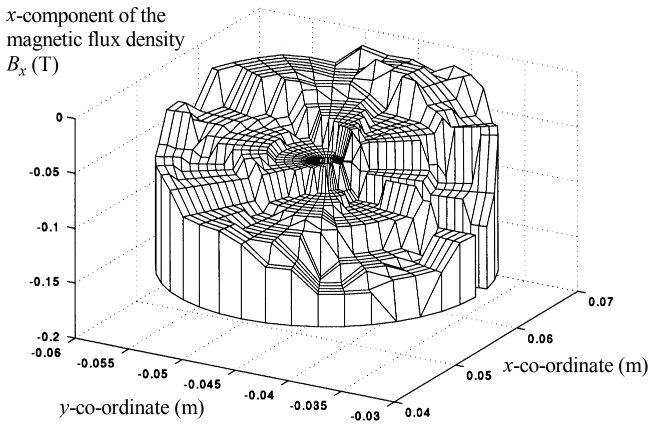


Fig. 3. Numerical calculation of the *x*-component of the magnetic flux density.

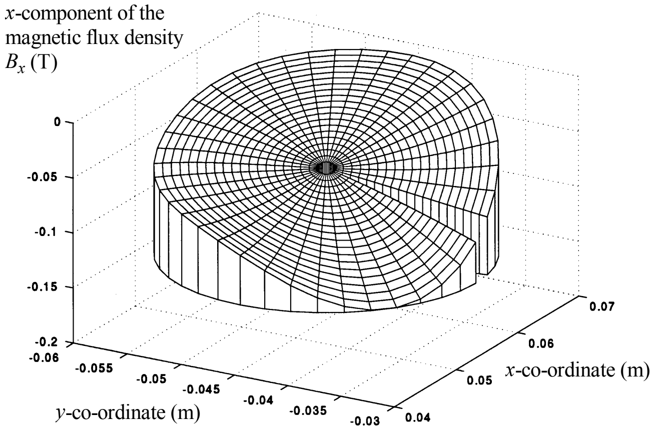
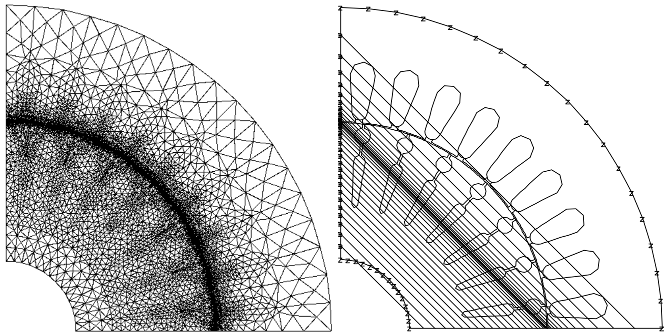


Fig. 4. Laplace based calculation of the *x*-component of the magnetic flux density.

Both methods can easily be extended to time-harmonic problems. If all values are rms-values the torque is obtained by adding the torque calculated using the real and the imaginary solution.

$$T = T_{real} + T_{imag} \tag{10}$$

Fig. 6 shows the variation of the torque for different values of the radius r of the contour. 1024 points are equidistantly spaced over the contour. The dashed line is the result of the conventional Maxwell stress method. The first contour is placed in the middle of the first layer of elements in the air gap, the last one in the middle of the third layer. The solid line is the result of the Laplace based torque calculation. The result is symmetric because two contours are needed to calculate the value of the torque. Fig. 6 clearly shows that the conventional Maxwell stress method strongly depends on the place of the contour in the air gap, while the Laplace based method gives similar values for the torque as long as both contours are placed in the middle layer of elements. Furthermore is the conventional Maxwell stress method sensitive to the uniformity of the finite elements in the air gap.



a) b)

Fig. 5. Finite element model and outline and constraint plot of the induction machine.

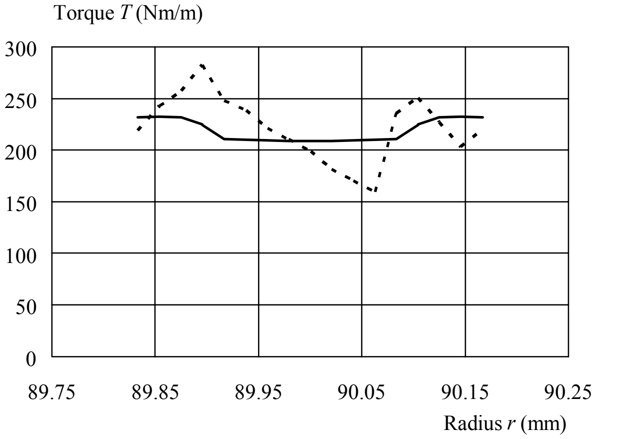


Fig. 6. Torque variation for different values of the radius r of the contour C.

IV. Conclusion

In combination with lower order finite elements, a local solution of Laplace's equation results in more accurate local field values for a given computation time. A higher accuracy of force and torque is obtained using these field values in the Maxwell stress method.

References

[1] K.J. Binns, P.J. Lawrenson, C.W. Trowbridge, *The Analytical and Numerical Solution of Electric and Magnetic Fields*, John Wiley & Sons, 1992.

[2] M. Kasper, J. Franz, "Highly accurate computation of field quantities and forces by superconvergence in finite elements," *IEEE Transactions on Magnetics*, May 1995.

[3] P.P. Silvester and D. Omeragic, "Differentiation of finite element solutions of Poisson equation," *IEEE Transactions on Magnetics*, vol. 29, pp. 1993-1996, March 1993.

[4] K. Hameyer, U. Pahner and R. Belmans, "Enhanced accuracy of 2D finite element field quantities by a local post-processor," *Proc. conf. 3rd Int. Conf. on Computation in Electromagnetics*, Bath, UK, 1996.