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Computation of electromagnetic force densities: Maxwell stress tensor vs. virtual work principle[☆]

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Abstract

A couple of fundamental formulae are demonstrated in this paper, which allow a systematic algebraic derivation of local electromagnetic forces in any material, starting from the expression of the energy density of that material. The derivation can be achieved in terms of vector and tensor analysis notions exclusively, provided the distinction is properly made between fields that are ‘flux densities’ (like \mathbf{b}) or ‘circulation densities’ (like \mathbf{h}). Applying the procedure to the particular case of a nonmagnetic material, the Maxwell stress tensor of empty space and the virtual work principle based formula for nodal forces are both readily found back. This makes the link obvious between those methods. The formulae are further applied to a permanent magnet material.

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1. Introduction

The idea behind the definition of electromagnetic (EM) forces is rather simple: they are given by the variation of the EM energy of the system when a configuration parameter is modified and the EM field is kept constant. Although this definition, which is basically a partial derivative in a properly defined variable set, is easy to state, the different steps to its implementation in a finite element (FE) programme remain quite obscure and uncertain. Moreover, there exist two distinct

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families of EM force formulae, depending on whether they are based on the *Maxwell stress tensor* or on the application of the *virtual work principle*. This distinction further contributes to confuse the issue.

A look into the literature shows that the expression of the Maxwell stress tensor (see e.g., [13]) is commonly obtained by algebro-differential operations starting from Maxwell equations. On the other hand, the nodal force formulae obtained by application of the virtual work principle rely on a FE mesh and are computed by a cumbersome roundabout way involving the jacobian matrix of coordinate transformation [1,5,6,10,11]. In both cases, coordinates are used and the fundamental thermodynamic concepts are buried into an overwhelming algebra. The issue has also been treated in a coordinate-free manner in [2,3], but at the expense of resorting to the more involved mathematical framework of *differential geometry*. Another observation is that all these approaches disregard the role played by the underlying matter. They assume a priori a specific expression for the magnetic constitutive law but fail to ask the fundamental question: How is this law affected by deformation? Consequently, applicability conditions of classical formulae are unclear, and hardly interpretable in the context of a new material. The blind test consisting in the numerical confrontation of different formulae has therefore been quite a popular game [7–9,12].

2. Definition of EM forces in a continuous medium

Whereas it sounds obvious to anybody that one can freely modify the magnetic field in a system without deforming it, by increasing the imposed currents for instance, it is much less clear to imagine how the system could be deformed without modifying the magnetic field. One feels indeed that any deformation of the system will have an effect on the magnetic field. The first step towards any analysis of an electromechanical system is therefore to define the magnetic and mechanical *states* in such a way that they are independent of each other.

Let first M be the *material manifold*, i.e. a continuous set of points representing each a material particle of a given electromechanical system. Let $C_1(M)$ be the set of all regular curves in M and $C_2(M)$ be the set of all regular surfaces. Let E be the Euclidean space \mathbb{R}^3 . Following [4], the magnetic state of the electromechanical system is defined by the *magnetic flux map*

$$\phi : C_2(M) \mapsto \mathbb{R}, \quad (1)$$

which associates a real number, the *magnetic flux*, to any surface in M . Similarly, the kinematics of the system is defined by the *placement map*

$$p : M \mapsto \Omega \subset E, \quad (2)$$

which associates to any point of M its position in E .

Although the magnetic flux map (1) determines completely the fluxes in the system, it does not give the local value of the *induction field*. In this context, *fields* are considered as secondary quantities obtained by a kind of interpolation. The properties of that interpolation process are not trivial. In the case of the induction field, the selection of a set of facets of $C_2(M)$ is involved, as well as an accuracy and convergence analysis. The interpolation with Whitney facet elements is an example of such an interpolation. Another example will be given below. Therefore, the induction field is noted $b(\phi, p)$, as it may depend on both p and ϕ . This dependency is the reason why the

interpolated induction field b is not a suitable primary variable in a thermodynamic representation. Similarly, a *position field* is associated with the placement map. It is noted $x(p)$.

Since the maps ϕ and p are independent of each other, they are suitable variables for the definition of the *energy functional* of the electromechanical system Ω . One has

$$\Psi(\phi, p) = \int_{\Omega(p)} \varrho^{\Psi}(b(\phi, p), x(p)) \, d\Omega, \tag{3}$$

where the *energy density* $\varrho^{\Psi}(b, x)$ depends on the maps ϕ and p through the *interpolated fields* b and x . If the problem is more easily posed in terms of the magnetic field h than in terms of b , the available state function is the *coenergy functional*

$$\Phi(I, p) = \int_{\Omega(p)} \varrho^{\Phi}(h(I, p), x(p)) \, d\Omega, \quad I : C_1(M) \mapsto \mathbb{R}, \tag{4}$$

where I is the *magnetomotive force map*, which associates a real number, the *magnetomotive force*, to any curve in M . The *definition of forces* follows now from the variation $\delta\Psi(\phi, p)|_{\delta\phi=0}$ or $\delta\Phi(I, p)|_{\delta I=0}$ of those energy functionals and the factorization of the result thereof under the form of a mechanical work monomial (e.g., $\int_{\Omega} \rho^f \cdot \delta\mathbf{x} \, d\Omega$, $\int_{\Omega} \sigma : \varepsilon \, d\Omega$, $\mathbf{f} \cdot \delta\mathbf{x}, \dots$).

It was assumed so far that Φ and Ψ were the *total (co)energy* functions of the system. Their differentiation therefore yields *total forces*, i.e. forces of mechanical *and* electromagnetic nature together. However, in the algebraic expression of the *total (co)energy* function, some terms are rather of an electromagnetic origin (even if they involve mechanical variables as well) and some others rather of a mechanical origin. In that sense, it can be split up, in a more or less significant way, into the sum of an electromagnetic and a mechanical term, i.e. $\Phi(\phi, p) = \Phi_e(\phi, p) + \Phi_m(p, \phi)$. The *electromagnetic force* is then defined as the one obtained by differentiation with respect to p of the first term only, but it must be kept in mind that this definition is somewhat arbitrary. The only quantity that really makes sense is the total force.

3. Application at the local level

We are looking for an intrinsic theory, i.e. a coordinate-free formulation. In the particular case of an Euclidean 3D space, *vector analysis* provides the needed intrinsic notions (e.g., vector, cross product, dot product, gradient, etc.). Notions of *tensor analysis* (e.g., dyadic product, tensor product, etc.) are also added in order to complete the mathematical framework. Let us now consider as *material manifold* M a unit cube with coordinates $\{\alpha, \beta, \gamma\}$ (Fig. 1).

Let $O \equiv (0, 0, 0)$, $A \equiv (1, 0, 0)$, $B \equiv (0, 1, 0)$ and $C \equiv (0, 0, 1)$ be four particular points of M . The position field $\mathbf{x}(p)$ is defined by the affine combination¹ of the placement of those four points

$$\mathbf{x} = (1 - \alpha - \beta - \gamma) p(O) + \alpha p(A) + \beta p(B) + \gamma p(C). \tag{5}$$

This determines in E a parallelepiped region Ω of volume

$$V = (\mathbf{r} \times \mathbf{s}) \cdot \mathbf{t} = (\mathbf{s} \times \mathbf{t}) \cdot \mathbf{r} = (\mathbf{t} \times \mathbf{r}) \cdot \mathbf{s}, \tag{6}$$

¹ A linear combination of which the sum of the coefficients equals 1.

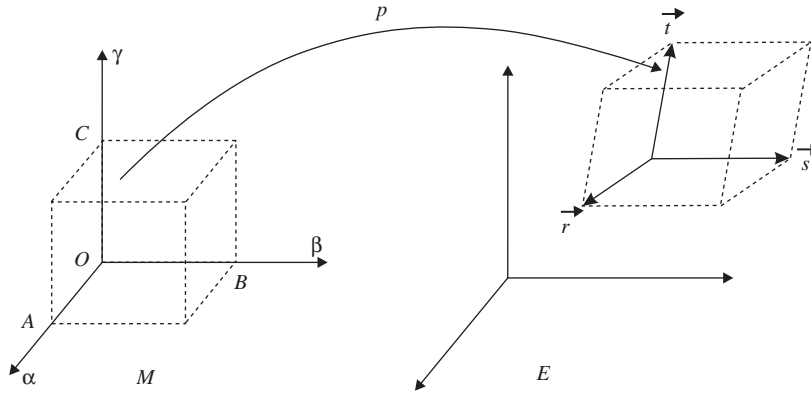


Fig. 1. Theoretical setup for the electromechanical coupling in a continuous medium.

where $\mathbf{r} = p(A) - p(O)$, $\mathbf{s} = p(B) - p(O)$ and $\mathbf{t} = p(C) - p(O)$ are three linearly independent vectors of E . By the properties of the mixed product, one has

$$AA^{-1} \equiv \begin{pmatrix} \mathbf{r} \\ \mathbf{s} \\ \mathbf{t} \end{pmatrix} \begin{pmatrix} \frac{\mathbf{s} \times \mathbf{t}}{V} & \frac{\mathbf{t} \times \mathbf{r}}{V} & \frac{\mathbf{r} \times \mathbf{s}}{V} \end{pmatrix} = \mathbb{I}, \tag{7}$$

where the A is the matrix of which the three lines are the vectors \mathbf{r} , \mathbf{s} and \mathbf{t} , and \mathbb{I} is the identity tensor. One can then check the important formula

$$V\mathbb{I} = VA^{-1}A = (\mathbf{s} \times \mathbf{t})\mathbf{r} + (\mathbf{t} \times \mathbf{r})\mathbf{s} + (\mathbf{r} \times \mathbf{s})\mathbf{t}, \tag{8}$$

where the dyadic (undotted) product is defined by $(\mathbf{ab})_{ij} = a_i b_j$. Taking the gradient of (5), one gets $\nabla \mathbf{x} \equiv \mathbb{I} = \nabla \alpha \mathbf{r} + \nabla \beta \mathbf{s} + \nabla \gamma \mathbf{t}$, which by identification with (8), gives

$$\nabla \alpha = \frac{\mathbf{s} \times \mathbf{t}}{V}, \quad \nabla \beta = \frac{\mathbf{t} \times \mathbf{r}}{V}, \quad \nabla \gamma = \frac{\mathbf{r} \times \mathbf{s}}{V}. \tag{9}$$

If the parallelepiped Ω is deformed by perturbing the placement of the points O , A , B and C , the displacement field and its gradient are:

$$\mathbf{u} \equiv \delta \mathbf{x} = (1 - \alpha - \beta - \gamma) \delta p(O) + \alpha \delta p(A) + \beta \delta p(B) + \gamma \delta p(C), \tag{10}$$

$$\nabla \mathbf{u} = \nabla \alpha \delta \mathbf{r} + \nabla \beta \delta \mathbf{s} + \nabla \gamma \delta \mathbf{t}, \quad \text{trace}(\nabla \mathbf{u}) = (\nabla \mathbf{u}) : \mathbb{I} = \frac{\delta V}{V}. \tag{11}$$

3.1. Formulation in b

Let ϕ_α , ϕ_β and ϕ_γ be the fluxes obtained by applying the magnetic flux map (1) to the particular parallelogram facets OBC , OCA and OAB of M . If the parallelepiped is small enough, the induction field in Ω is the uniform vector field \mathbf{b} that verifies

$$\phi_\alpha = (\mathbf{s} \times \mathbf{t}) \cdot \mathbf{b}, \quad \phi_\beta = (\mathbf{t} \times \mathbf{r}) \cdot \mathbf{b}, \quad \phi_\gamma = (\mathbf{r} \times \mathbf{s}) \cdot \mathbf{b}. \tag{12}$$

By inverting this relation,² one finds the expression for this case of the interpolated induction field $\mathbf{b}(\phi, p)$:

$$\mathbf{b} = \frac{\mathbf{r}}{V} \phi_\alpha + \frac{\mathbf{s}}{V} \phi_\beta + \frac{\mathbf{t}}{V} \phi_\gamma. \tag{13}$$

The parallelepiped Ω is now deformed by perturbing the placement of the point C , i.e., by perturbing the vector \mathbf{t} , leaving \mathbf{r} and \mathbf{s} unchanged. The gradient of the displacement field is in this case

$$\nabla \mathbf{u} = \frac{\mathbf{r} \times \mathbf{s}}{V} \delta \mathbf{t}, \quad \text{trace}(\nabla \mathbf{u}) = \frac{\mathbf{r} \times \mathbf{s}}{V} \cdot \delta \mathbf{t}. \tag{14}$$

The variation of \mathbf{b} with fluxes held constant is

$$\delta \mathbf{b}|_{\delta \phi=0} = -\frac{\delta V}{V^2} (\mathbf{r} \phi_\alpha + \mathbf{s} \phi_\beta + \mathbf{t} \phi_\gamma) + \frac{\delta \mathbf{t}}{V} \phi_\gamma. \tag{15}$$

Once the variation is done, it is allowed to substitute back for \mathbf{b} . Using (12), (13) and (6), one finds

$$\delta \mathbf{b}|_{\delta \phi=0} = \frac{1}{V} \{-\mathbf{b}(\mathbf{r} \times \mathbf{s}) \cdot \delta \mathbf{t} + \mathbf{b} \cdot (\mathbf{r} \times \mathbf{s}) \delta \mathbf{t}\} \tag{16}$$

and finally, using (11),

$$\delta \mathbf{b}|_{\delta \phi=0} = \mathbf{b} \cdot \nabla \mathbf{u} - \mathbf{b} \text{trace}(\nabla \mathbf{u}). \tag{17}$$

If Ω is made of a nonmagnetic material, the energy function (3) and its variation with fluxes held constant are

$$\Psi = V \frac{|\mathbf{b}|^2}{2\mu_0}, \quad \delta \Psi|_{\delta \phi=0} = \frac{V}{\mu_0} \mathbf{b} \cdot \delta \mathbf{b}|_{\delta \phi=0} + \delta V \frac{|\mathbf{b}|^2}{2\mu_0}. \tag{18}$$

Using (11), (17) and the property $\mathbf{a} \cdot (\nabla \mathbf{u}) \cdot \mathbf{b} = (\mathbf{a} \mathbf{b}) : (\nabla \mathbf{u})$, it can be written

$$\delta \Psi|_{\delta \phi=0} = V \nabla \mathbf{u} : \sigma_M(\mathbf{b}) \quad \text{with} \quad \sigma_M(\mathbf{b}) = \frac{1}{\mu_0} \left(\mathbf{b} \mathbf{b} - \frac{|\mathbf{b}|^2}{2} \mathbb{I} \right), \tag{19}$$

which is the expression of the *Maxwell stress tensor* [13] associated with a nonmagnetic material.

3.2. Formulation in h

Let I_r , I_s and I_t be the circulations obtained by applying the magnetomotive force map (4) to the edges OA , OB and OC of M . If the parallelepiped Ω is small enough, the magnetic field in Ω is the uniform vector field that verifies

$$I_r = \mathbf{r} \cdot \mathbf{h}, \quad I_s = \mathbf{s} \cdot \mathbf{h}, \quad I_t = \mathbf{t} \cdot \mathbf{h}, \tag{20}$$

hence, by inversion, the interpolated magnetic field $\mathbf{h}(I, p)$:

$$\mathbf{h} = \frac{\mathbf{s} \times \mathbf{t}}{V} I_r + \frac{\mathbf{t} \times \mathbf{r}}{V} I_s + \frac{\mathbf{r} \times \mathbf{s}}{V} I_t. \tag{21}$$

² Note that (12) can be written $\phi = VA^{-T} \mathbf{b}$ and premultiply both sides by A^T .

The variation of \mathbf{h} with currents held constant is

$$\delta\mathbf{h}|_{\delta I=0} = -\frac{\delta V}{V}\mathbf{h} + \frac{1}{V}\{\mathbf{s} \times \delta\mathbf{t} I_t + \delta\mathbf{t} \times \mathbf{r} I_s\} = -\frac{1}{V}(\mathbf{r} \times \mathbf{s})\mathbf{h} \cdot \delta\mathbf{t} \tag{22}$$

by (20) and using the variation of (8), which writes as well

$$\delta\mathbf{h}|_{\delta I=0} = -\nabla\mathbf{u} \cdot \mathbf{h}. \tag{23}$$

Considering again a nonmagnetic material, the coenergy function (4) and its variation with currents held constant gives are

$$\Phi = V\mu_0 \frac{|\mathbf{h}|^2}{2}, \quad \delta\Phi|_{\delta I=0} = V\mu_0\mathbf{h} \cdot \delta\mathbf{h}|_{\delta I=0} + \delta V\mu_0 \frac{|\mathbf{h}|^2}{2} \tag{24}$$

which factorizes as

$$\delta\Phi|_{\delta I=0} = -V\nabla\mathbf{u} : \sigma_M(\mathbf{h}) \quad \text{with} \quad \sigma_M(\mathbf{h}) = \mu_0 \left(\mathbf{h}\mathbf{h} - \frac{|\mathbf{h}|^2}{2} \mathbb{1} \right), \tag{25}$$

which is again the Maxwell stress tensor of empty space, although the intermediary steps were somewhat different.

Note how different are the expressions of the ‘flux density field’ \mathbf{b} (13) and of the ‘circulation density field’ \mathbf{h} (21), although they are commonly considered as being both vector fields of the same nature. Their behaviour under deformation is thus different as well, as seen by comparing (17) and (23). On the other hand, the Maxwell stress tensors (19) and (25) are independent of the particular placement of M . They have therefore a *local* meaning and they can be used as an applied stress in a structural analysis.

4. The Coulomb technique

Local EM forces formulae can be found by a direct application of the virtual work principle in a FE context [1,5,6,10,11]. Those formulae can be found back straightforwardly by applying the procedure described in the previous section to a *tetrahedral* material manifold M (Fig. 2).

The idea is now to factorize the variation of the EM (co)energy functionals under the form of the force–displacement monomial: $\delta\Psi|_{\delta\phi=0} = \mathbf{f} \cdot \delta\mathbf{x}$. The only differences are a few scalar factors, due to the fact that the facets are halved and the volume of the tetrahedron is $V' = V/6$. According to (14) and (9), one has $\nabla\mathbf{u} = (\nabla\gamma)\delta\mathbf{t}$ and one finally finds

$$\delta\Psi|_{\delta\phi=0} = \mathbf{f} \cdot \delta\mathbf{t}, \quad \mathbf{f} = V'(\nabla\gamma)\sigma_M, \quad \mathbf{f}_j = \frac{V'}{\mu_0} \frac{\partial\gamma}{\partial x_i} \left(b_i b_j - \frac{|\mathbf{b}|^2}{2} \delta_{ij} \right). \tag{26}$$

This is equivalent to the formulae in [6,11] if one notes that γ can play the role of the barycentric first-order nodal shape function in a deforming tetrahedral finite element, i.e., γ is a linear function of x that is zero at all vertices but the tip of \mathbf{t} . It should be stressed that this EM force \mathbf{f} is *not* local as it involves explicitly γ and V' , which depend on the particular placement of the tetrahedron.

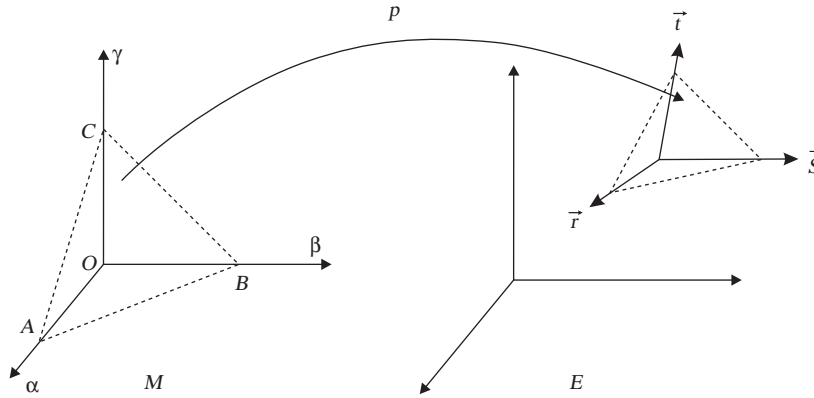


Fig. 2. Theoretical setup with a tetrahedral material manifold.

5. Examples

Eqs. (17) and (23) are the fundamental formulae to derive the Maxwell stress tensor and the force density from the expression of the energy density in any material. Applying them to complex (ferromagnetic, magnetostrictive, etc.) materials would first require an in-depth thermodynamic analysis of the material’s behaviour, which is not the scope of this paper. However, by way of example, the table hereunder gives the expression obtained for a few simple magnetic material models.

	ρ_{EM}^ψ	σ_{EM}	ρ_{EM}^f
1	$\frac{ \mathbf{b} ^2}{2\mu}$	$\frac{\mathbf{b}\mathbf{b}}{\mu} - \parallel \frac{ \mathbf{b} ^2}{2\mu}$	$\text{curl} \frac{\mathbf{b}}{\mu} \times \mathbf{b}$
2	$\frac{ \mathbf{b} ^2}{2\mu} - \mathbf{b} \cdot \mathbf{m}$	$\frac{\mathbf{b}\mathbf{b}}{\mu} - \parallel \frac{ \mathbf{b} ^2}{2\mu}$	$\text{curl} \left(\frac{\mathbf{b}}{\mu} - \mathbf{m} \right) \times \mathbf{b} + \text{curl} \mathbf{m} \times \mathbf{b}$
3	$\frac{ \mathbf{b} ^2}{2\mu} - \mathbf{b} \cdot \mathbf{m}$	$\frac{\mathbf{b}\mathbf{b}}{\mu} - \mathbf{b}\mathbf{m} - \mathbf{m}\mathbf{b} - \parallel \left(\frac{ \mathbf{b} ^2}{2\mu} - \mathbf{b} \cdot \mathbf{m} \right)$	$\text{curl} \left(\frac{\mathbf{b}}{\mu} - \mathbf{m} \right) \times \mathbf{b} + \mathbf{m} \times \text{curl} \mathbf{b}$
4	$\frac{ \mathbf{b} - \mu\mathbf{m} ^2}{2\mu}$	$\frac{\mathbf{b}\mathbf{b}}{\mu} - \mu\mathbf{m}\mathbf{m} - \parallel \left(\frac{ \mathbf{b} ^2}{2\mu} - \frac{\mu \mathbf{m} ^2}{2} \right)$	$\text{curl} \left(\frac{\mathbf{b}}{\mu} - \mathbf{m} \right) \times \mathbf{b} + \text{curl} \mathbf{m} \times (\mathbf{b} - \mu\mathbf{m})$

The first line recalls the expressions found in case of a linear material ($\mu = cte$). The second line gives the expressions found for a permanent magnet, if the constant magnetisation \mathbf{m} is assumed to be a ‘circulation density’ like \mathbf{h} . On the contrary, the third line gives the expressions found for a permanent magnet, if the magnetisation is now assumed to be a ‘flux density’ like \mathbf{b} . One sees that these two assumptions lead to different expressions for the EM forces. Finally, the fourth line assumes again that \mathbf{m} is a ‘circulation density’, but the energy density has been chosen so as to be zero at the remanence point ($\mathbf{h} = 0$) instead of at the demagnetised state ($\mathbf{b} = 0$). This expression

of the energy density give the same magnetic constitutive law (i.e., $\mathbf{h} = \mathbf{b}/\mu - \mathbf{m}$), but it leads to a different expression for the EM force.

6. Comments

One has shown that the appropriate level to tackle with the theoretical definition of local EM forces is the level of an infinitesimal but finite box (parallelepiped or tetrahedron). At the limit, the *Maxwell stress tensor* is found to be the fundamental local expression of the electromechanical coupling in a continuous medium. It can be directly used as an applied stress in a structural analysis. The local *EM force density*, i.e. the divergence of the Maxwell stress tensor $\rho_M^f = \nabla \cdot \sigma_M$, is by this approach an outcome rather than a postulate.

The description of the electro-mechanical coupling is completely contained in the chosen expression of the (co)energy density of the material. Each material has therefore its own expression of the Maxwell stress tensor, which can be found by using the fundamental formulae (17) and (23). In the process, new questions arise (e.g., what is the geometrical nature of the magnetisation, what is the zero of the EM energy density functional) which were irrelevant to the EM model, but which are well relevant to the electro(magneto-)mechanical coupled model. It is a matter of measurements and experiments to answer those questions.

The classical notion of continuous placement map (see e.g., [2,4]) is not the one we use here. The position field x (5) is not here defined as the image of p . It is rather defined as the combination of two elements: an affine combination (which ensures the continuity of matter) and the placement of a minimal set of characteristic points (which describes the deformation of the body in a finite dimension space). This splitting is the cornerstone of the proposed method.

7. Conclusion

A couple of fundamental formulae have been demonstrated, which allow a systematic algebraic derivation of local electromagnetic forces. The proposed approach is intrinsic and all written in terms of classical *vector* and *tensor analysis*. Nevertheless, it owes to *differential geometry* the fundamental distinction between ‘flux densities’ and ‘circulations densities’. The link between the Maxwell stress tensor and the virtual work principle has been drawn. The meaning of the expression “derivation with fluxes (or currents) held constant”, which is used when applying the virtual work principle, has been clarified and given a theoretical ground. The proposed formulae are applicable to any material provided the expression of the (co)energy density is given by a prior thermodynamic analysis of the material.

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