Electromagnetic Force Density in a Ferromagnetic Material

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***Abstract*— Material derivatives for electromagnetic fields are presented. They are applied to determine the Maxwell stress tensor associated with different models of ferromagnetic materials. The accent is placed on the theoretical implications of the simplifying assumptions done in the different material models, with in scope a thermodynamically consistent finite-element implementation of electromagnetic forces.**

***Index Terms*— Electromagnetic forces, electromechanical effects, ferromagnetic materials, magnetomechanical effects, magnetostriction, modeling, permanent magnets.**

I. Introduction

IT is well-known that electromagnetic (EM) fields generate forces and stresses. In literature, forces and torques exerted on pointwise EM sources (charges, dipoles, moments, ...) are often considered as the fundamental expressions of the electromechanical coupling. Energies for systems of such pointwise sources are indeed easy to express (by summation of two-by-two contributions) and the force exerted on a given particle is easily obtained by a simple derivation (gradient) with respect to the coordinates of that particle.

But the finite-element method requires to represent materials as continuous media. Distribution of pointwise EM sources have thus to be translated into densities. This translation is generally done in a formal way, each pointwise quantity being simply replaced by its corresponding density (charge density, ...). This way of working disregards the possible interactions between pointwise sources, and ignores the material structure. Moreover, when one tries to translate pointwise forces into their continuous equivalent, severe mathematical difficulties arise (problem of the self-force, integral of singular kernels, inconsistencies, ... [1]) which are technical and give no insight into the physics involved.

It is therefore desirable to avoid that detour by pointwise fictitious quantities and to define forces directly from a continuous medium representation of the materials. The mathematical analysis of this problem requires to consider a deforming body, and to apply adequately energy conservation rules to it. The correct background to perform such operations is differential geometry (See e.g. [2]), and one needs in particular the Lie derivative $\pounds_{\mathbf{v}}$, where $\mathbf{v}$ denotes the velocity field. Fortunately, the final results of the analysis can be expressed in the language of vector analysis (See [6], or [4], [5] for more classical approach with a control volume). This gives in section II a set of formulae, which must be considered as axioms. With these formulae, the Maxwell stress tensor (MST) of any material can be derived straightforwardly from the expression of the EM energy density of that material. The Maxwell stress tensor is thus, according to this energy-based approach, the fundamental representation of the electromechanical coupling in a continuous medium.

The purpose of this paper is to apply the proposed formulae to different models of ferromagnetic material (reversible case). In each case, an energy functional is proposed, the main features of the associated constitutive laws are drawn and finally the expression of the MST is derived.

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II. Material derivatives for EM fields

Differential geometry provides the rules to compute the material derivative $\mathcal{L}_{\mathbf{v}} = \partial_t + \pounds_{\mathbf{v}}$, of any tensor field. The particular tensor fields we need in this paper are the *differential forms* [2], [3]. In a three dimensional space, there exist 4 kinds of differential forms called $p-$forms, $p = 0, 1, 2, 3$. Their material derivatives are

$$\mathcal{L}_{\mathbf{v}} f = \dot{f} \quad (1)$$

$$\mathcal{L}_{\mathbf{v}} \mathbf{h} = \dot{\mathbf{h}} + (\nabla \mathbf{v}) \cdot \mathbf{h} \quad (2)$$

$$\mathcal{L}_{\mathbf{v}} \mathbf{b} = \dot{\mathbf{b}} - \mathbf{b} \cdot (\nabla \mathbf{v}) + \mathbf{b}\ \mathrm{tr}(\nabla \mathbf{v}) \quad (3)$$

$$\mathcal{L}_{\mathbf{v}} \rho = \dot{\rho} + \mathrm{tr}(\nabla \mathbf{v})\ \rho = \partial_t \rho + \mathrm{div}\,(\rho \mathbf{v}) \quad (4)$$

respectively for the $0-$forms (which are the scalar fields), the $1-$forms (e.g. the magnetic field), the $2-$forms (e.g. the induction field) and the $3-$forms (which are the density fields). In (1 - 4), $\dot{z}$ denotes the *total derivative* of $z(t, x^k)$, obtained by applying the chain rule of derivatives, component by component in fixed Euclidean axes if z is a vector field. The tensor $(\nabla \mathbf{v})_{ij} \equiv \partial_i v_j$ is the gradient of the velocity field, whose trace is $\mathrm{tr}(\nabla \mathbf{v}) \equiv \partial_i v_i \equiv \mathrm{div}\, \mathbf{v}$. Equations (1) and (4) are used in fluid dynamics to express conservation laws.

III. Linear material

Throughout the paper, the density of a quantity X will be denoted ρ^X. One has thus

$$X = \int_\Omega \rho^X \quad , \quad \dot{X} \equiv \int_\Omega \mathcal{L}_{\mathbf{v}} \rho^X = \int_\Omega (\dot{\rho}^X + \mathrm{tr}(\nabla \mathbf{v})\ \rho) \quad (5)$$

by definition of the material derivative [2] and (4). Note the extra term accounting for the deformation of Ω. In case of a linear material ($\mu = cte$), the magnetic energy density is $\rho^\Psi = \frac{1}{2\mu}|\mathbf{b}|^2$ and the variation of the magnetic energy in a volume Ω is

$$\dot{\Psi} = \int_\Omega \left(\frac{\mathbf{b}}{\mu} \cdot \dot{\mathbf{b}} + \frac{|\mathbf{b}|^2}{2\mu} \mathrm{tr}(\nabla \mathbf{v}) \right). \quad (6)$$

In terms of the material derivative, the work W_{EM} done by the EM forces has a clear definition: $W_{EM} \equiv \dot{\Psi}\Big|_{\mathcal{L}_{\mathbf{v}}\mathbf{b}=0}$.

Equation (3) is used to substitute for $\dot{\mathbf{b}}$ in (6), and one gets

$$W_{EM} = \int_\Omega \left(\frac{\mathbf{b}}{\mu} \cdot (\nabla \mathbf{v}) \cdot \mathbf{b} - \frac{|\mathbf{b}|^2}{2\mu} \mathrm{tr}(\nabla \mathbf{v}) \right), \qquad (7)$$

which factorizes into a stress-strain product

$$W_{EM} = \int_\Omega \nabla \mathbf{v} : \sigma_M \quad , \quad \sigma_M = \frac{1}{\mu} \left(\mathbf{b}\ \mathbf{b} - \frac{|\mathbf{b}|^2}{2} \mathbb{I} \right) \qquad (8)$$

where $\mathbb{I}$ stands for the identity matrix. Note the use of the *dyadic* (undotted) *vector product* $(\mathbf{a}\ \mathbf{b})_{ij} = a_i b_j$, the inner tensor product $a : b = a_{ij} b_{ij}$ (implicit summation assumed) and the property $(\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) = (\mathbf{a}\ \mathbf{b}) : (\mathbf{c}\ \mathbf{d})$.

Similarly, in terms of the magnetic field, the coenergy density is $\rho^\Phi = \frac{1}{2}\mu |\mathbf{h}|^2$ and one has

$$\dot{\Phi}\Big|_{\mathcal{L}_{\mathbf{v}}\mathbf{h}=0} = -\int_\Omega \nabla \mathbf{v} : \sigma_M \ , \ \sigma_M = \mu \left(\mathbf{h}\ \mathbf{h} - \frac{|\mathbf{h}|^2}{2} \mathbb{I} \right). \qquad (9)$$

In both cases the classical expressions of the Maxwell stress tensor of free space ($\mu = \mu_0$) [7] are found back.

IV. Permanent Magnet

The energy density of a permanent magnet material is

$$\rho^\Psi = \frac{|\mathbf{b}|^2}{2\mu_0} - \mathbf{b} \cdot \mathbf{m}, \qquad (10)$$

where the magnetization $\mathbf{m}$ does not depend on $\mathbf{b}$. One has however a modelisation choice to make. If one decides to represent the magnetization as a *circulation density* ($1-$form), one finds out that the term $\mathbf{b} \cdot \mathbf{m}$ in (10) gives no contribution to the Maxwell stress tensor. Indeed, substituting for $\dot{\mathbf{b}}$ and $\dot{\mathbf{m}}$ using respectively (3) and (2), one finds

$$\left(\frac{d}{dt} \int_\Omega \mathbf{b} \cdot \mathbf{m} \right) \Big|_{\mathcal{L}_{\mathbf{v}}\mathbf{b}=0, \mathcal{L}_{\mathbf{v}}\mathbf{m}=0} = 0. \qquad (11)$$

Under this assumption, the Maxwell stress tensor of the permanent magnet material is (8) with $\mu = \mu_0$. If one decides on the contrary to represent the magnetization as a *flux density* ($2-$form), one finds by using now (3) for $\dot{\mathbf{m}}$

$$\sigma_M = \frac{\mathbf{b}\ \mathbf{b}}{\mu_0} - \mathbf{b}\ \mathbf{m} - \mathbf{m}\ \mathbf{b} - \left(\frac{|\mathbf{b}|^2}{2\mu_0} - \mathbf{b} \cdot \mathbf{m} \right) \mathbb{I}, \qquad (12)$$

which is quite different from the former result.

This shows that the distinction between flux densities and circulation densities, which is irrelevant for the expression of the *magnetic* constitutive law, becomes essential when the *magneto-mechanical* coupling is considered. There is however no *mathematical* reason to favour one or the other of these assumptions. It is only a matter of experiment to determine case by case which model is the closest to reality.

V. Polycrystalline Material

In case of an polycrystalline ferromagnetic material, a classical expression of the EM energy density is

$$\rho^\Psi(\mathbf{b}, \varepsilon) = \int_0^{\mathbf{b}} \mathbf{h}(\mathbf{b}, \varepsilon) \cdot d\mathbf{b}, \qquad (13)$$

with ε the strain tensor. The idea is to collate measurements of $\mathbf{h}(\mathbf{b}, \varepsilon)$ and to integrate them in order to define an energy functional. This approach has however the following drawbacks. Firstly, the integral with vector bounds in (13) requires a mathematical definition. A suitable definition is

$$\rho^\Psi(\mathbf{b}, \varepsilon) = \int_0^t \mathbf{h}(\mathbf{b}_t(u), \varepsilon_t(u)) \cdot \dot{\mathbf{b}}_t(u)\ du \qquad (14)$$

where $\mathbf{b}_t(t)$ and $\varepsilon_t(t)$ represent paths starting from a reference state and verifying $\mathbf{b}_t(t) = \mathbf{b}$ and $\varepsilon_t(t) = \varepsilon$. The energy density is properly defined if the integral is independent of the chosen paths, which gives integrability conditions that are not fulfilled *a priori* by measurements. Secondly, an energy functional like (14) does not rely on a material model as it is directly built from measurements.

We shall therefore adopt a simpler model

$$\rho^\Phi = \int_0^{|\mathbf{h}|} \mu_0 \left(1 + \chi(x)\right) x\ dx, \qquad (15)$$

where the *magnetic susceptibility* χ is allowed to depend on the modulus of $\mathbf{h}$ only, which gives an unambiguously defined integral. As there is no explicit dependency of χ with strain, this material model does not take magnetostriction into account, but it gives a good representation of isotropic polycrystalline saturable materials, like iron and non-laminated steel. The Maxwell stress tensor associated with (15) is

$$\sigma_M = \mu_0 (1 + \chi(\mathbf{h}))\ \mathbf{h}\ \mathbf{h} - \rho^\Phi \mathbb{I}. \qquad (16)$$

We shall see in section VII how a material model with magnetostriction can be built.

VI. Monocrystal

Ferromagnetic crystals are substances with permanent atomic magnetic moments that are coupled by exchange forces of quantum-mechanical origin. Therefore, they line up with each other and create an intense magnetic moment density. On the other hand, in order to minimize the overall magnetostatic energy, a macroscopic ferromagnetic monocrystal tends to break up into several domains, called Weiss domains. The magnetic moment density is homogeneous within each domain but its orientation varies from one domain to the next, so that the total magnetic moment of the sample vanishes if no external field is applied. Because of anisotropy and provided the material is not highly saturated, the magnetic moments of the domains are preferably oriented along one of the easy-axes of magnetization of the crystal. These are the mechanisms we are going to implement in a simple but realistic material model. Energy of Bloch walls and slight deviations of the moments from the easy-axes are not considered here.

A small piece Ω of a ferromagnetic monocrystal is considered at an intermediary scale between the microscopic structure of the ferromagnet (Weiss domains, ...) and the

characteristic macroscopic dimensions of the system. Let m_s be the saturation magnetization of that monocrystal. A 2D monocrystal is considered, with two easy-axes of magnetization, respectively aligned with the $\mathbf{r}$ and $\mathbf{s}$ directions. There are then 4 possible states for each atomic moment, i.e. $\pm\mathbf{r}$ and $\pm\mathbf{s}$, Fig. 1. Let a, b, c, d be their relative populations, with $a, b, c, d \in [0, 1]$ and $a + b + c + d = 1$.

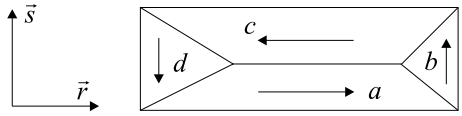

Fig. 1. A simple ferromagnetic monocrystal material model.

The magnetic coenergy in Ω is

$$\Phi = \int_\Omega \mu_0 \left(\frac{|\mathbf{h}|^2}{2\mu_0} - \mathbf{h} \cdot \mathbf{m} + \Gamma \right) \tag{17}$$

with

$$\mathbf{m} = m_s \left((a - c)\, \mathbf{r} + (b - d)\, \mathbf{s} \right), \tag{18}$$

$$\Gamma = C \frac{m_s^2 \xi^2}{2}, \tag{19}$$

$$\xi^2 = (b - a)^2 + (c - b)^2 + (d - c)^2 + (a - d)^2. \tag{20}$$

The first term in (17) is the magnetostatic energy and the second one is the potential energy of the magnetic moments in the applied magnetic field. The third one modelizes the long-range interaction between domains. It is interpreted as follows. As shown at Fig. 1, the four domains can be considered as forming a magnetic circuit. Each imbalance between two successive sections of that circuit creates a kind of leakage field, which increases the magnetostatic energy. As it stands for a long range effect, Γ can reasonably be assumed not to depend on the strain.

The EM coenergy functional (17) depends on the variables $\mathbf{h}$ and $a_k = \{a, b, c, d\}$. For a given $\mathbf{h}$, the magnetization is determined by the constrained minimization

$$a_k^\star(\mathbf{h}) \equiv \min_{a_k} \Phi(\mathbf{h}, a_k) \tag{21}$$

with respect to the *internal variables* a_k. Fig. 2 shows the evolution of the populations a, b, c, d for one cycle of an applied sinusoidal $\mathbf{h}$ field.

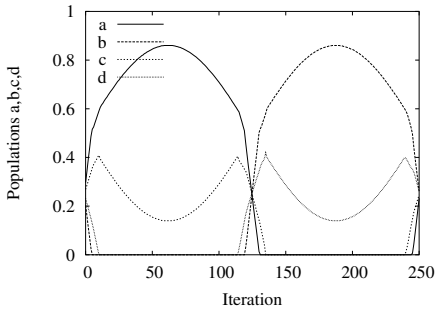

Fig. 2. Evolution of the populations a, b, c, d for one cycle of the applied $\mathbf{h}$ field, making an angle $22.5°$ with the $\mathbf{r}-$axis.

This simple model of a ferromagnetic monocrystal, which involves only two free parameters, m_s and C, is yet able to represent saturation and anisotropy, as shown at Fig. 3. The value of C is related with the susceptibility of the material, i.e. the slope at zero-field of the magnetization curve. In general, the value of the free parameters are determined by matching the constitutive laws derived from the material model with measurements. For a better accuracy of the model, more elaborated expressions of ξ can be considered.

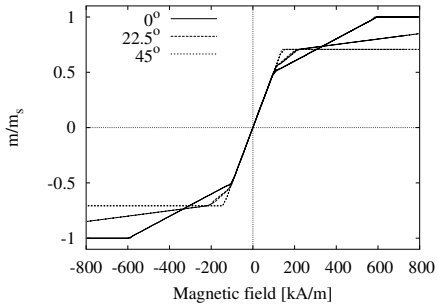

Fig. 3. Magnetization curves obtained for different directions of the applied $\mathbf{h}$ field, with $C = 0.25$.

Forces are again obtained by considering the variation of the magnetic coenergy. The a_k's being internal variables and not fields, their derivative $\dot{a}_k$ is a simple time derivative. The variation of the EM energy is thus evaluated with $\mathcal{L}_\mathbf{v}\mathbf{h} = 0$, $\mathcal{L}_\mathbf{v}\mathbf{m} = 0$ and $\dot{a}_k = 0$. The expression of the MST associated with this monocrystal model turns out to be identical to the one of a permanent magnet, since Γ will play no role in the magnetomechanical coupling. This means that it is more the existence of a magnetization in the material, than the way this magnetization depends on the magnetic field, that determines the magnetomechanical behaviour.

Fig. 3 shows also that the monocrystal model behaves isotropic and linear at low fields. We have thus three candidate models to represent linear isotropic materials at low fields: the linear model of section III and the two permanent magnet models of section IV. There are therefore also three candidate expressions of the MST: let model A be (8), model B be (8) with $\mu = \mu_0$ and model C be (12). If those three expressions are implemented and the deformation of a closed rectangular magnetic circuit is computed, one observes that the three models, which are perfectly equivalent on the magnetic side, give different deformations, Fig. 4. Again, it is a matter of experiment and modelling to determine, case by case, which model matches the best reality.

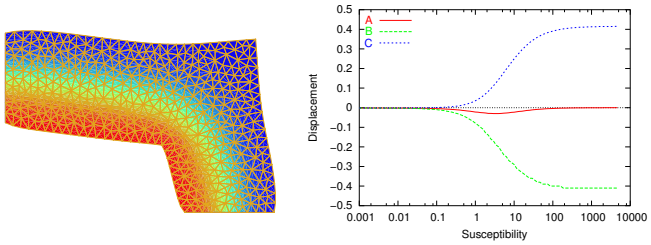

Fig. 4. On the left, a deformed state of a quarter of the magnetic core. On the right, displacement of the upper right corner as a function of the susceptibility χ of the linear material: $\mu = \mu_0(1 + \chi)$ for model A and $\mathbf{m} = \frac{\chi}{\mu_0(1+\chi)}\mathbf{b}$ for model B and model C.

VII. Magnetostrictive material

This quantitative model describes the magnetostrain effect observed in several ferromagnetic shape memory alloys [8]. The material is treated as a composite of three martensitic phases aligned with the crystallography axes [100], [010] and [001]. Let $x, y, z \in [0,1]$, $x+y+z=1$, be the respective populations of the three phases. For the phase aligned with [100], a spontaneous elongation ε_0 along the x-axis is observed and a contraction $-\varepsilon_0/2$ along the other two axes. On the other hand, for an applied magnetic field h_x along the x-direction, a magnetization $m_a(h_x)$ along the x-direction is observed and a slower magnetization $m_t(h_x)$ in the other two directions, Fig.5 and 6. As it goes similarly for the other two phases, the magnetization $\mathbf{m}$ and the strain ε can be written as:

$$\mathbf{m} \equiv M \cdot \mathbf{x} = \begin{pmatrix} m_a(h_x) & m_t(h_x) & m_t(h_x) \\ m_t(h_y) & m_a(h_y) & m_t(h_y) \\ m_t(h_z) & m_t(h_z) & m_a(h_z) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

$$\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{pmatrix} = \varepsilon_0 \begin{pmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Taking the relations $x+y+z=1$ and $\varepsilon_{xx}+\varepsilon_{yy}+\varepsilon_{zz}=0$ into consideration, this relation can be inverted into

$$\mathbf{x}(\varepsilon) \equiv \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \frac{2}{3\varepsilon_0}\begin{pmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{zz} \end{pmatrix}. \qquad (22)$$

The EM coenergy density of this material is

$$\rho^{\Phi} = \mu_0 \left(\int_0^{\mathbf{h}} d\mathbf{h} \cdot M \cdot \mathbf{x}(\varepsilon) + \frac{|\mathbf{h}|^2}{2} \right). \qquad (23)$$

Since $(\dot{\varepsilon}_{xx}, \dot{\varepsilon}_{yy}, \dot{\varepsilon}_{zz})^T = \mathrm{diag}(\nabla \mathbf{v})$, it leads to the MST

$$\sigma_M = \mu_0 \mathbf{h}\ (M \cdot \mathbf{x} + \mathbf{h}) + \frac{2\mu_0}{3\varepsilon_0} \int_0^{\mathbf{h}} d\mathbf{h} \cdot M \cdot \mathbb{J} - \rho^{\Phi}\mathbb{I} \quad (24)$$

where $\mathbb{J}$ is the tensor defined by $\mathbb{J}_{ijk} = 1$ if $i=j=k$ and $\mathbb{J}_{ijk} = 0$ otherwise.

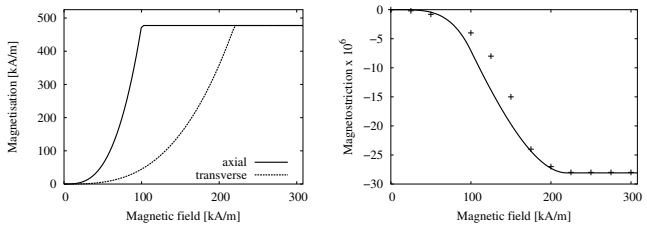

Fig. 5. On the left, the magnetization curves $m_a(h)$ (axial) and $m_t(h)$ (transverse). On the right, the corresponding magnetostriction. The points represent measured values for pure Ni [9].

VIII. Conclusion

The purpose of this paper is to place theoretical landmarks for the finite element modelling of electromechanical interactions in materials. An energy-based approach has been chosen so that the model is necessarily consistent from the thermodynamical point of view and the reciprocity conditions are automatically fulfilled. In particular, the Maxwell stress tensor of the material can be systematically derived by applying the formulae of the material derivative of EM fields.

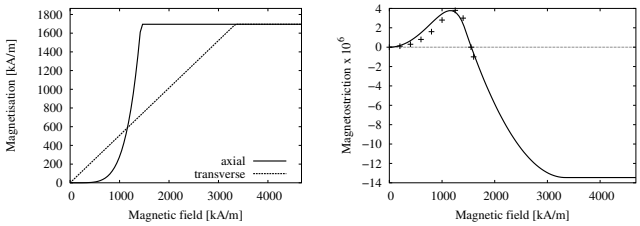

Fig. 6. On the left, the magnetization curves $m_a(h)$ (axial) and $m_t(h)$ (transverse). On the right, the corresponding magnetostriction. The points represent measured values for pure Fe [9].

The Maxwell stress tensor finds so its natural place in the energy-based description of materials, i.e. the partial derivative of the electromagnetic energy with respect to the cinematic variables. As the material derivatives for the EM fields involve the *gradient* of the velocity field, the Maxwell stress tensor σ_M (and not the EM force density $\rho^{\mathbf{f}}_{EM} \equiv \mathrm{div}\,\sigma_M$), is the fundamental expression of the electromechanical coupling in a continuous medium. All other expressions of forces and stresses can be deduced from it. The Maxwell stress tensor can be used directly as an applied stress in the structural equation $\mathrm{div}\,(\sigma+\sigma_M)+\rho^{\mathbf{f}}=0$. The *resultant force* on a moving body is obtained by integrating it over an enclosing surface. Finally, the classical nodal force formulae proposed in e.g. [10], [11], can be readily found back by multiplying it with the gradient of a nodal shape function and integrating over the support of that shape function, see [12], [5].

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