THE EGGSHELL APPROACH FOR THE COMPUTATION OF ELECTROMAGNETIC FORCES IN 2D AND 3D

François Henrotte, Geoffrey Deliége, Kay Hameyer *
Katholieke Universiteit Leuven, Dept. ESAT, Div. ELECTA
Kasteelpark Arenberg 10, B-3001 Heverlee-Leuven, Belgium
e-mail: francois.henrotte@esat.kuleuven.ac.be

ABSTRACT

The expressions of the Lie derivative of differential forms in the language of vector analysis are introduced. These formulae allow to describe naturally the electromechanical coupling, and the coupling term appears to be a volume integral. A general approach to compute forces is then proposed, which takes that fact into consideration. The method is applicable in 2D and in 3D, and with dual formulations. Numerical evidences of its efficiency are given.

I. INTRODUCTION

The existence of such a long controversy about the computation of electromagnetic (EM) forces is undoubtedly to ascribe to the fact that the problem cannot be solved with the tools of vector analysis. The mathematical analysis of this problem requires indeed to consider a deforming body, and to apply adequately energy conservation rules to it. The correct background to perform such operations is differential geometry (See e.g. [1]), and one needs in particular the Lie derivative. Fortunately, the final results of the analysis can be expressed in the language of vector analysis. This gives in section II a set of formulae, which must be considered as axioms, and are used in section III to solve the problem of the electromechanical coupling in a continuous medium. It turns out that the fundamental representation of the electromechanical coupling term has the form of a stress-strain product, where the Maxwell stress tensor plays by definition the role of the stress. This leads in section IV to a new approach for the computation of EM forces, which is more clearly backed by the theory.

II. LIE DERIVATIVE AND MATERIAL DERIVATIVE

Let M be a continuous set of points and $u_t(X), X \in M, t \in [a, b]$ be the trajectory of point X in an euclidean space E. The set of trajectories of all points in M defines a flow. We call *placement* the map

$$p_t \, : X \in M \mapsto u_t(X) \in E, t \in [a, b]. \tag{1}$$

The flow, which is entirely defined by the placement map, is assumed to be smooth and regular enough to be differentiable and invertible when required.

The *velocity* $\mathbf{v}$ at point $x = u_t(X)$ is the vector tangent to the curve $u_t(X)$. It is defined by $\mathbf{v} = \frac{\partial}{\partial t} u_t(X)$ and belongs to T_xE, the set of all vectors anchored at point x. The *velocity field* is the set of tangent vectors to all trajectories of the flow at a given instant of time.

The notions of length and angle are defined in E by means of the *metric*

$$g \, : \mathbf{v}, \mathbf{w} \in T_xE \mapsto g(\mathbf{v}, \mathbf{w}) = g_{ij} v^i w^j \in \mathbb{R} \tag{2}$$

which, at each point x, associates a number to any pair of anchored vectors. An euclidean space is characterised by $g_{ij} = \delta_{ij}$.

Let us now consider a small piece of curve in E. As each point of the curve follows its own trajectory, the curve deforms, i.e. it changes in length, orientation, curvature, etc. But the so-called *vectors*, which are by definition the vectors tangent to all curves in E, are also transformed by the flow, and so is it as well in general for all tensors. All required information to describe that transformation, called convection, is actually contained in the placement map p_t. So a tensor field T becomes $p_{t+dt}(p_t^{-1} T)$ at time $t + dt$ by the only effect of flow convection. If now $T \neq p_{t+dt}(p_t^{-1} T)$, the tensor field has got a non-zero derivative along the flow. The *Lie derivative* of the tensor field $\pounds_{\mathbf{v}} T$ [1] is precisely that derivative along the flow. It is defined by

$$\pounds_{\mathbf{v}} T = \lim_{dt \to 0} \frac{p_t(p_{t+dt}^{-1} T) - T}{dt}. \tag{3}$$

Finally, if the tensor field T depends also on time, the *material derivative* is defined by :

$$\mathcal{L}_{\mathbf{v}} T = \frac{\partial T}{\partial t} + \pounds_{\mathbf{v}} T, \tag{4}$$

where a notation with the velocity field explicitly mentioned has been prefered in order to remind that the material derivative depends on the flow.

Differential geometry provides the rules to compute the Lie derivative and the material derivative of any tensor field, and in particular of the *differential forms* [1, 2], which are the particular tensor fields we need in this paper. In a three dimensional space, there exist 4 kinds of differential forms called $p-$forms, $p = 0, 1, 2, 3$, which all have a specific expression of the material derivative, i.e.

$$\mathcal{L}_{\mathbf{v}} f = \frac{\partial f}{\partial t} + v^k \frac{\partial f}{\partial x^k} \tag{5}$$

$$(\mathcal{L}_{\mathbf{v}} \mathbf{h})_i = \frac{\partial h_i}{\partial t} + v^k \frac{\partial h_i}{\partial x^k} + \frac{\partial v^k}{\partial x^i} h_k \tag{6}$$

$$(\mathcal{L}_{\mathbf{v}} \mathbf{b})^i = \frac{\partial b^i}{\partial t} + v^k \frac{\partial b^i}{\partial x^k} - b_k \frac{\partial v^i}{\partial x^k} + b^i \frac{\partial v^k}{\partial x^k} \tag{7}$$

$$\mathcal{L}_{\mathbf{v}} \rho = \frac{\partial \rho}{\partial t} + v^k \frac{\partial \rho}{\partial x^k} + \rho \frac{\partial v^k}{\partial x^k} \tag{8}$$

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respectively for the $0-$forms (e.g. a scalar function), the $1-$forms (e.g. the magnetic field), the $2-$forms (e.g. the induction field) and the $3-$forms (e.g. the energy density). With obvious definitions, this can be written with more concise notations

$$\mathcal{L}_{\mathbf{v}} f = \dot{f} \tag{9}$$
$$\mathcal{L}_{\mathbf{v}} \mathbf{h} = \dot{\mathbf{h}} + (\nabla \mathbf{v}) \cdot \mathbf{h} \tag{10}$$
$$\mathcal{L}_{\mathbf{v}} \mathbf{b} = \dot{\mathbf{b}} - \mathbf{b} \cdot (\nabla \mathbf{v}) + \mathbf{b}\, \mathrm{tr}(\nabla \mathbf{v}) \tag{11}$$
$$\mathcal{L}_{\mathbf{v}} \rho = \dot{\rho} + \mathrm{tr}(\nabla \mathbf{v})\, \rho \tag{12}$$

where $\dot{z}$ denotes the *total derivative* of $z(t, x^k)$, obtained by applying the chain rule, component by component if z is a vector field. Finally, the material derivative allows to compute the time derivative of integrals over moving domains:

$$\frac{d}{dt} \int_{\Omega} \rho \, d\Omega = \int_{\Omega} \mathcal{L}_{\mathbf{v}} \rho \, d\Omega. \tag{13}$$

III. MAXWELL STRESS TENSOR

In an electromechanical problem, the variation of the EM energy *functional* is not equal to the variation (in the sense of change) of the EM energy stored in the system. One misses indeed the work W_{EM} done by the EM forces. Let the *EM energy density* ρ^{Ψ} of an electromechanical system Ω be a known function of the induction field $\mathbf{b}$. By means of the formulae (9-13) and the classical chain rule of derivatives, the time derivative of the EM energy Ψ writes

$$\begin{aligned}
\dot{\Psi} &= \int_{\Omega} \mathcal{L}_{\mathbf{v}} \rho^{\Psi} = \int_{\Omega} \left(\dot{\rho}^{\Psi} + \mathrm{tr}(\nabla \mathbf{v})\, \rho^{\Psi} \right) \\
&= \int_{\Omega} \left(\frac{\partial \rho^{\Psi}}{\partial \mathbf{b}} \cdot \dot{\mathbf{b}} + \mathrm{tr}(\nabla \mathbf{v})\, \rho^{\Psi} \right) \\
&= \int_{\Omega} \left(\frac{\partial \rho^{\Psi}}{\partial \mathbf{b}} \cdot \mathcal{L}_{\mathbf{v}} \mathbf{b} \right) \\
&+ \int_{\Omega} \left(\mathbf{b} \cdot \nabla \mathbf{v} \cdot \frac{\partial \rho^{\Psi}}{\partial \mathbf{b}} - \mathrm{tr}(\nabla \mathbf{v}) \left(\frac{\partial \rho^{\Psi}}{\partial \mathbf{b}} \cdot \mathbf{b} - \rho^{\Psi} \right) \right).
\end{aligned} \tag{14}$$

The first term at the r.h.s. (14) is the *definition* of the change in stored EM energy and the second term is the mechanical power $\dot{W}_{EM}$ *received* by the EM system.

A similar calculation for the *EM coenergy* Φ gives

$$\begin{aligned}
\dot{\Phi} &= \int_{\Omega} \left(\frac{\partial \rho^{\Phi}}{\partial \mathbf{h}} \cdot \mathcal{L}_{\mathbf{v}} \mathbf{h} \right) \\
&- \int_{\Omega} \left(\frac{\partial \rho^{\Phi}}{\partial \mathbf{h}} \cdot \nabla \mathbf{v} \cdot \mathbf{h} - \mathrm{tr}(\nabla \mathbf{v})\, \rho^{\Phi} \right)
\end{aligned} \tag{15}$$

with here the first term at the r.h.s. the change in stored EM coenergy and the second term $-\dot{W}_{EM}$.

One can now notice that $\dot{W}_{EM}$ does not involve the velocity field $\mathbf{v}$ itself but only its gradient $\nabla \mathbf{v}$. The *Maxwell stress tensor* is by definition the dual of the latter:

$$\dot{W}_{EM} = \int_{\Omega} \sigma_{EM} : \nabla \mathbf{v}. \tag{16}$$

Simple calculations give

$$\sigma_{EM} = \mathbf{b} \frac{\partial \rho^{\Psi}}{\partial \mathbf{b}} - \left(\frac{\partial \rho^{\Psi}}{\partial \mathbf{b}} \cdot \mathbf{b} - \rho^{\Psi} \right) \mathbb{I} \tag{17}$$
$$\sigma_{EM} = \frac{\partial \rho^{\Phi}}{\partial \mathbf{h}} \mathbf{h} - \rho^{\Phi} \mathbb{I} \tag{18}$$

where $\mathbb{I}$ is the identity matrix, resp. for the formulations in $\mathbf{b}$ and in $\mathbf{h}$. Note the use of the dyadic (undotted) vector product $(\mathbf{v}\, \mathbf{w})_{ij} = v^i w^j$ and the tensor product $a : b = a_{ij} b_{ij}$.

It should be carefully noted that the Maxwell stress tensor σ_{EM} is defined as a true mechanical stress, i.e. its work is delivered by the mechanical system and received by the electromagnetic system. On the other hand, the EM forces defined by $\rho^{\mathbf{f}} = \mathrm{div}\, \sigma_{EM}$ are magnetic forces. Their work is delivered by the electromagnetic system and received by the mechanical system. This should be clearer after integrating (16) by part:

$$\int_{\Omega} \sigma_{EM} : \nabla \mathbf{v} = - \int_{\Omega} \rho^{\mathbf{f}}_{EM} \cdot \mathbf{v} + \int_{\partial \Omega} \mathbf{n} \cdot \sigma_{EM} \cdot \mathbf{v} \tag{19}$$

with $\partial\Omega$ the boundary of Ω and $\mathbf{n}$ the exterior normal to $\partial\Omega$. Moreover, being defined as the EM energy dual of $\nabla \mathbf{v}$ at the local level, the Maxwell stress tensor can, as such, directly play the role of an applied stress in the structural equations of the system

$$\mathrm{div}\, (\sigma + \sigma_{EM}) + \rho^{\mathbf{f}} = 0, \tag{20}$$

which is easier than coupling through the EM forces $\rho^{\mathbf{f}}_{EM}$, since the latter are singular at material interfaces.

IV. THE EGGSHELL APPROACH

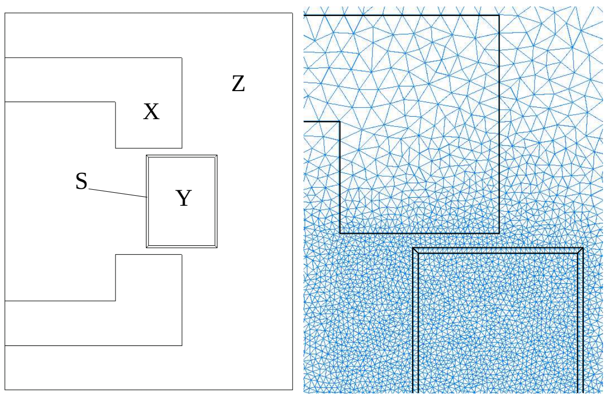

Figure 1: Geometry of the c-core and detail of the mesh in the airgap.

Let us consider a system Ω with a piece Y that can move in the aperture of a C-core X, not completely represented here. An eggshell shaped region S is defined, that encloses the moving piece (Fig. 1) and whose thickness need not be constant. The region Z is defined such that $X \cup Y \cup S \cup Z = \Omega$; Z and S only contain air. The problem is now how to compute the EM forces on Y. The natural mechanical unknowns of this problem are the velocities $\mathbf{v}$ (or equivalently the displacements) at all nodes of the region Y. We have seen however that the coupling term (16) involves a velocity field, virtual or not, defined on the whole study domain Ω. We must thus first understand the role played by the velocity field $\mathbf{v}$ in $\Omega - Y$. For the sake of simplicity, X and $\partial\Omega$ are assumed rigid and fixed, i.e. we are only interested in the forces on Y. We have then from (19)

$$\int_{\Omega - Y} \sigma_{EM} : \nabla \mathbf{v} = - \int_{\partial Y} \mathbf{n} \cdot \sigma_{EM} \cdot \mathbf{v}, \tag{21}$$

because $\mathbf{v} = 0$ on $X \cup \partial\Omega$ (clamped rigid parts) and $\rho^{\mathbf{f}}_{EM} = 0$ in $Z \cup S$ (air). This means that the contribution of the exterior of Y to the coupling term is completely determined by the value of the velocity field on its boundary ∂Y. Consequently, the velocity field $\mathbf{v}$ is arbitrary in the *interior* of $Z \cup S$, but it must connect continuously with $\mathbf{v}$ on ∂Y, which is not zero. The velocity field blurs thus necessarily out of the moving region. The idea of the eggshell approach is to set the velocity field to zero in Z, confining the non-zero velocity field in the shell S, and of course in Y.

Let us now state that the moving piece Y is rigid and shifted by an infinitesimal amount $\delta\mathbf{u}$. The only region that deforms is S. The (virtual) velocity field associated with that deformation, and its gradient are

$$\mathbf{v} = \gamma\, \delta\dot{\mathbf{u}} \quad , \quad \nabla\mathbf{v} = \nabla\gamma\, \delta\dot{\mathbf{u}}, \tag{22}$$

where γ is any smooth function whose value is 1 on the inner surface of the shell and 0 on the outer surface. Using (16), one can write

$$\dot{W}_{EM} = -\mathbf{F} \cdot \delta\dot{\mathbf{u}} = \int_S \sigma_{EM} : \nabla\mathbf{v}\; dS, \tag{23}$$

where $\mathbf{F}$ is the *resultant force* on Y, and finally, using (22), one gets

$$\mathbf{F} = -\int_S \sigma_{EM} \cdot \nabla\gamma\; dS, \tag{24}$$

which is the eggshell formula for the EM resultant force on a rigid body. Only the Maxwell stress tensor of empty space is here required. The formula applies in 2D and in 3D. It applies also directly to dual formulations, provided one uses (17) for the $\mathbf{b}-$formulation and (18) for the $\mathbf{h}-$formulation. The eggshell formula can be seen as a generalized variant of Coulomb's technique to compute nodal EM forces [3, 4] and of Arkkio's formula for torque in electrical machines [5]. At the limit for an infinitely thin shell, one finds back the classical result that the resultant EM force on a rigid body is given by the flux of the Maxwell stress tensor through an enclosing surface.

The eggshell formula for rigid body movement is tested in 2D on the C-core problem (Fig. 1). The moving piece Y (3 mm by 4 mm) is inserted in the magnetic core X, leaving an airgap of 0.4 mm on both sides. The magnetic horizontal force tends to bring the moving piece back in alignment with the C-core. The problem is solved with dual finite element formulations, so as to check the accuracy of the computed fields and forces, Fig. 2. The constitutive law $\mathbf{b} = \mu(|\mathbf{h}|)\mathbf{h}$ with

$$\mu(h) = \begin{cases} a + \mu_{fix} & \text{if } h \leq h_{fix} \\ a + \frac{1}{d\,h + c} & \text{if } h > h_{fix} \end{cases} \tag{25}$$

with $c = 1/\mu_{fix} - d\; h_{fix}$, is representative of a saturable material and has the technical advantage that it can be inverted, i.e. $\mathbf{h}$ can be expressed as a function of $\mathbf{b}$, and the (co)energy functionals can be integrated analytically. The parameters were set to $\mu_{fix} = 7.55\,10^{-3}$, $h_{fix} = 103.35$, $a = 1.5\,10^{-5}$ and $d = 0.625$, (all quantities in SI units).

In Fig. 3, the global forces computed with the eggshell formula (24) are compared with the forces computed by a direct differentiation of the EM (co)energy, using a second order finite difference scheme for the derivative. A perfect match is observed, which shows the validity of the eggshell approach. The eggshell formula however, requires only one solution of the system whereas direct differentiation requires several solutions, with slightly changed positions of the moving body. The difference between the values computed with the $\mathbf{b}-$formulations and with the $\mathbf{h}-$formulation are due to the discretisation error. For variational consistency, it is better not to mix fields from different formulations when evaluating the Maxwell stress tensor, i.e. for instance, not to mix the $\mathbf{h}$ field from a $\mathbf{h}-$formulations with the $\mathbf{b}$ field from a $\mathbf{b}-$formulations, although this may seem a good idea from the point of view of the individual accuracy of the different fields. Fig. 4 shows indeed that the forces computed with the mixed expression $\sigma_{EM} = \mathbf{b}\,\mathbf{h} - \frac{\mathbf{b}\cdot\mathbf{h}}{2}\mathbb{I}$ are less accurate.

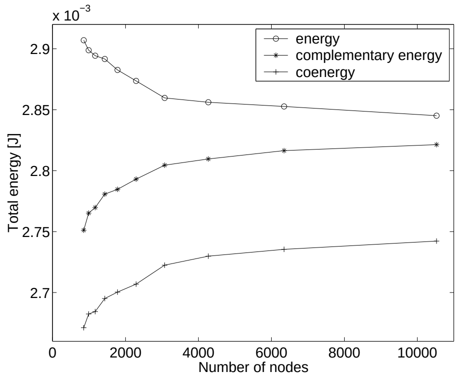

Figure 2: Energy Ψ, coenergy Φ and complementary energy $\int_\Omega \mathbf{b} \cdot \mathbf{h}\; d\Omega - \Phi$ as a function of the number of nodes. The difference between energy and complementary energy is a measure of the global discretisation error.

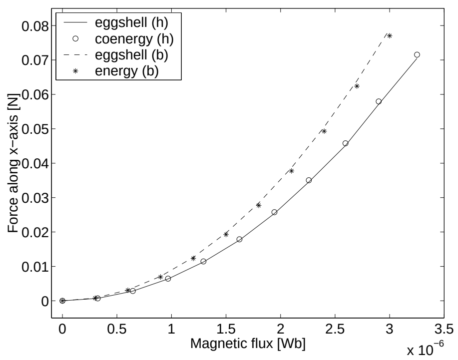

Figure 3: Comparison of the horizontal forces computed with the eggshell method and the direct derivation of energy ($\mathbf{b}-$formulation) or of coenergy ($\mathbf{h}-$formulation).

The eggshell approach gives a certain freedom in the definition of the shell. This is one of its advantages. The shape is actually free and the shell needs not be in contact with the moving piece. The effect of the thickness of the shell and of the distance between the moving piece and the shell are shown at Fig. 5 and Fig. 6 respectively. One sees that a better accuracy is obtained if the shell is not placed directly in contact with the magnetic moving piece, because of the singularity of EM fields at material corners. Another way to define the eggshell is to select all finite elements in $\Omega - Y$ that have at least one node on ∂Y. The γ function is then the sum of the shape functions of the nodes of ∂Y.

This way of defining the eggshell has been used in 3D to compute the deformation of a rectangular magnetic frame, of which by symmetry only one quarter was modelled, Fig. 7.

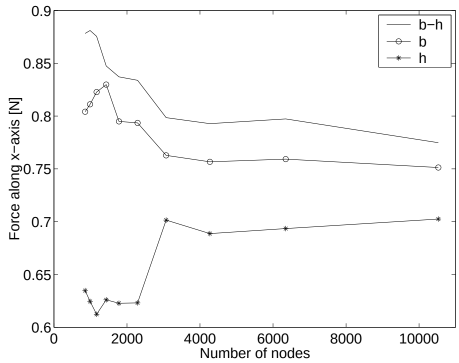

Figure 4: Effect of a variational inconsistency.

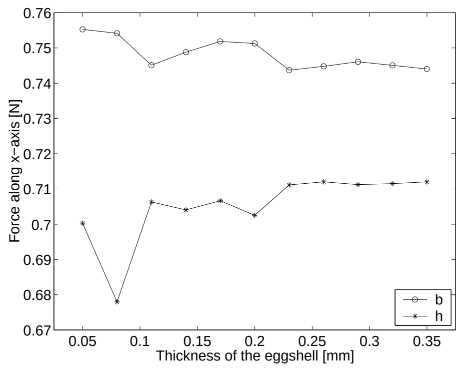

Figure 5: Effect of the thickness of the shell.

Let Y be the deforming piece. The weak form of (20) can be written

$$\int_Y \sigma : \nabla \mathbf{v}' + \int_\Omega \sigma_{EM} : \nabla \mathbf{v}' + \int_Y \rho^{\mathbf{f}}.\mathbf{v}' = 0 \quad \forall \mathbf{v}' \quad (26)$$

so as to make explicit use of the coupling term (16). As the trial functions $\mathbf{v}'$ are the shape functions of the nodes of Y, the integration of the coupling term can be limited to $Y \cup S$, where the egsshell S is the set of all finite elements in $\Omega - Y$ that have at least one node on ∂Y. In this case, the eggshell approach allows a very straightforward implementation of an electromechanical problem. It avoids to compute the trace of σ_{EM} on ∂Y, making benefit of the existing magnetic mesh outside the deforming piece.

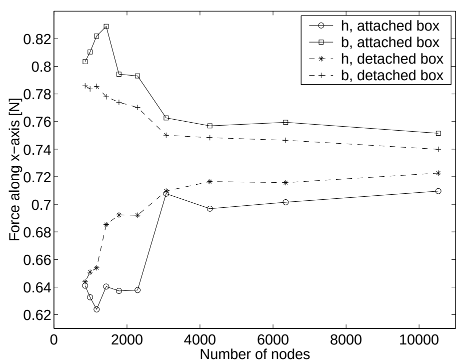

Figure 6: Effect of not placing the shell in contact with the ferromagnetic moving piece.

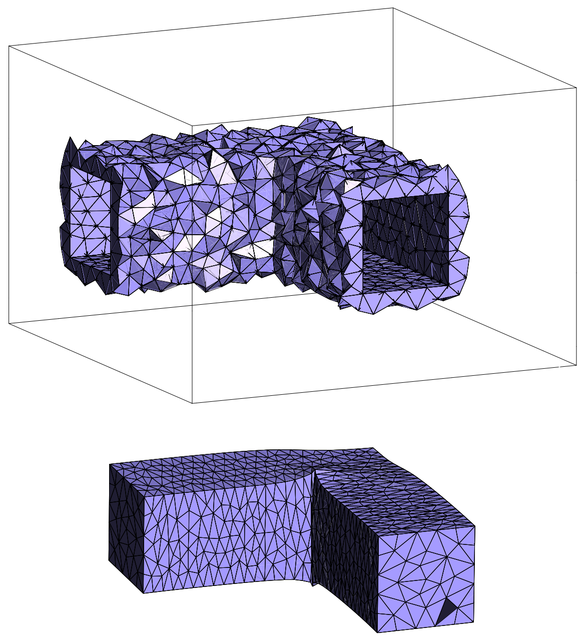

Figure 7: Eggshell around a quarter of the rectangular magnetic frame, and deformed state.

V. CONCLUSION

The Lie derivative of differential forms have been introduced in the language of vector analysis. They allow to determine the fundamental form of the electromechanical coupling term in continuous media. The eggshell approach is based on that particular form and the classical methods to compute EM forces are particular cases of it. However, the eggshell approach is more directly and more clearly linked with the underlying energy considerations at the continuous and at the discrete level, for rigid and non-rigid movements. This makes this approach easier to understand and to implement in a finite element program.

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