Electromagnetic force density in a ferromagnetic material

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***Abstract*— A very simple but realistic model for ferromagnetic materials is proposed under the form of a functional to minimise. This functional is then used to derive the corresponding expression of the Maxwell stress tensor.**

I. Introduction

In a recent paper, a systematic derivation of the electromagnetic (EM) force density from the expression of the EM energy density was presented by the authors. It was applied to air (no magnetisation) and permanent magnet materials (magnetisation is independent of the magnetic field). In this paper, it is intended to make one step further and to apply the same procedure to saturable ferromagnetic materials (reversible case).

As the forces are directly derived from the knowledge of the EM energy density, the problem is simply to find a satisfactory expression of the latter. The classical expression is

$$\rho^{\Psi}(\vec{b},\varepsilon) = \int_0^{\vec{b}} \vec{h}(\vec{b},\varepsilon)\cdot d\vec{b}, \tag{1}$$

which has the nice *formal* property that

$$\delta\rho^{\Psi}\big|_{\delta\varepsilon=0} = \vec{h}\cdot\delta\vec{b}. \tag{2}$$

The idea is then to collate measurements of $\vec{h}(\vec{b},\varepsilon)$ and to integrate them. This approach has however the following drawbacks.

Firstly, it does not rely on a real material model. The expression of the EM energy density is on the contrary *induced* from empirical constitutive laws, i.e. which are obtained from measurements. But measurements gives only a partial view on the complexity of the electromechanical behaviour of a real material, because of the limited number of measurements and the unavoidable experimental errors. Moreover, only *global quantities* (magnetic fluxes, displacements) are measured, whereas the constitutives laws are in terms of the corresponding *local quantities* ($\vec{b}$ and ε), and the required transposition from 'global' to 'local' is not trivial in practice and full of pitfalls and simplifying assumptions.

Secondly, the integral with vector-valued bounds in (1) requires a mathematical definition in order to be useable. A suitable definition is

$$\rho^{\Psi}(\vec{b},\varepsilon) = \int_0^{t} \vec{h}(\vec{b}_t(u),\varepsilon_t(u))\cdot\dot{\vec{b}}_t(u)\ du, \tag{3}$$

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This text presents research results of the Belgian programme on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming.

where $\vec{b}_t(t)$ and $\varepsilon_t(t)$ represent pathes starting from a reference state and verifying $\vec{b}_t(t)=\vec{b}$ and $\varepsilon_t(t)=\varepsilon$. The energy density is properly defined if the integral in (3) is independent of the chosen pathes, which gives integrability conditions that are not easy to fulfill in practice.

II. Ferromagnetic material

For these reasons, a completely different approach is proposed in this paper. Instead of basing the material description upon constitutive laws, as is usually done, we shall define a simplified but realistic model for the ferromagnetic materials.

Ferromagnetic materials are substances with permanent atomic magnetic moments that are coupled by long-range exchange forces of quantum-mechanical origin. Therefore, they line up with each other and create an intense magnetic moment density. On the other hand, in order to minimize the overall magnetic energy, a macroscopic ferromagnetic sample tends to break up into several domains, called Weiss domains. The magnetic moment density is homogeneous within each domain but its orientation varies from one domain to the next, so that the total magnetic moment of the sample vanishes if no external field is applied. Because of anisotropy and provided the material is not highly saturated, the magnetic moments of the domains are preferably oriented along one of the easy-axes of magnetisation of the crystal. These are the mechanisms we are going to implement in our material model.

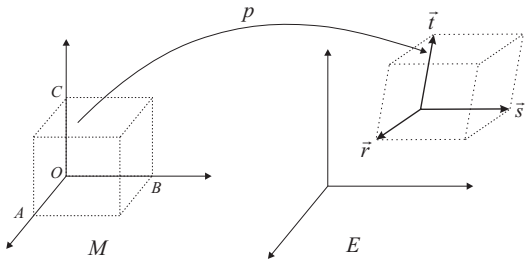

Fig. 1. Euclidean placement of a small piecce of ferromagnetic material.

A small piece of ferromagnetic material M is considered at an intermediary scale between the microscopic structure of the ferromagnet (Weiss domains, ...) and the characteristic macroscopic dimensions of the system. The placement map p determines in the euclidean space E a paralelepiped region $p(M)$ spanned by the vectors $\vec{r}$, $\vec{s}$ and $\vec{t}$. The volume of $p(M)$ is $V=(\vec{r}\times\vec{s})\cdot\vec{t}$. Let N be the number of magnetic moments in M and $\mathcal{M}_0$ their magnitude. These are constant numbers. Let us assume a 2D monocrystal, with two easy-axes of magnetisation, respectively aligned with the α and β

directions. There are then 4 possible states for each atomic moment, i.e. $\pm\alpha$ and $\pm\beta$. Let a, b, c and d be their relative populations, with $a, b, c, d \in [0, 1]$ and $a+b+c+d = 1$. This has the pictorial representations showed at Fig. 2.

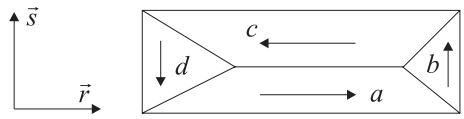

Fig. 2. A simple ferromagnetic materialmodel.

In order to calculate forces, one needs to express the induction field $\vec{b}$ in $p(M) \subset E$ explicitly as a function of the fluxes across the facets of M. Since the induction field is a *differential form of 2d degree*, one has in 2D (See [3], [4])

$$\vec{b}(\phi_i, \vec{r}_j) = \phi_{OBC}\frac{\vec{r}}{V} + \phi_{OCA}\frac{\vec{s}}{V}. \tag{4}$$

with ϕ_{OBC} and ϕ_{OCA} the fluxes across the facet OBC et OCA in Fig 1. On the other hand, if the $\rho^{\vec{\mathcal{M}}}$ is assumed to be a *differential form of 1st degree*, (See [4] for a discussion of this assumption), it is represented in E by

$$\rho^{\vec{\mathcal{M}}}(\vec{r}_j, a_k) = N\mathcal{M}_0 \left[(a-c)\ \frac{\vec{s}\times\vec{t}}{V} + (b-d)\ \frac{\vec{t}\times\vec{r}}{V}\right]. \tag{5}$$

Let us now define the *energy density* of our ferromagnetic material, subjected to an induction field $\vec{b}$, by

$$\rho^{\Psi} = \frac{|\vec{b}|^2}{2\mu_0} - \vec{b}\cdot\rho^{\vec{\mathcal{M}}} + C\frac{(N\mathcal{M}_0\xi)^2}{2} \tag{6}$$

with $\vec{b}$ given by (4), $\rho^{\vec{\mathcal{M}}}$ given by (5) and

$$\xi^2 = (b-a)^2 + (c-b)^2 + (d-c)^2 + (a-d)^2. \tag{7}$$

The first two terms in (6) are classical. The third one is a tentative expression for the *internal energy* of the sample. It is interpreted as follows. As shown at Fig. 2, the four domains can be considered as forming a magnetic circuit. Each imbalance between two successive moments in that circuit creates a kind of leakage field, which generates magnetostatic energy in the surroundings. This assumption is of course somewhat rudimentary, but we shall see it gives already very important and nicely general results.

III. Material model

The material model we are seeking for is given by the minimisation of the energy density functional (6). In order to get rid of the constraint $a+b+c+d = 1$, independent internal variables $\zeta, \eta, \theta \in [0, 1]$ are defined, such that $a = \zeta\ \eta$, $b = (1-\zeta)\ \theta$, $c = \zeta\ (1-\eta)$ and $d = (1-\zeta)\ (1-\theta)$. Equation (7) becomes

$$\xi^2 = 2 - 4\zeta^2\eta(1-\eta) - 4(1-\zeta)^2\theta(1-\theta) - 6\zeta(1-\zeta). \tag{8}$$

Using (8), the energy density functional ρ^{Ψ} can be written in terms of the *independent thermodynamic variables* $\phi_i, \vec{r}_j$ and ζ_k. The material model is now defined by the minimisation

$$\zeta_k^\star(\phi_i, \vec{r}_j) \equiv \min_{\zeta_k\in[0,1]} \rho^{\Psi}(\phi_i, \vec{r}_j, \zeta_k) \tag{9}$$

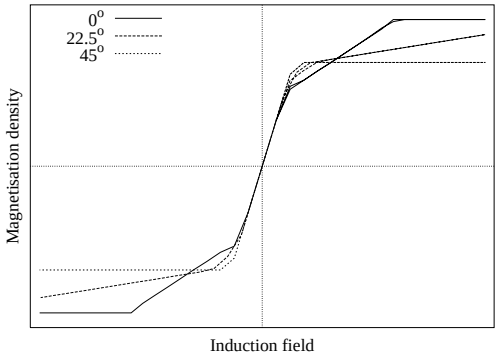

Fig. 3. Magnetisation curves as a function of the orientation of the magnetic field with respect to $\vec{r}$.

with respect to the *internal variables* ζ_k.

This simple model of a ferromagnetic material, which involves only one free parameter C, is already able to represent saturation and a certain kind of anisotropy, as shown at Fig. 3. In this case, the value of C is related with the susceptibility of the material, i.e. the slope at zero-field of the magnetisation curve. In general, the value of the free parameters are determined by matching the constitutive laws derived from the material model with measurements.

IV. Electromagnetic forces

The advantage of having a real material model is that it directly gives an expression for the energy density, from which forces are directly derived by

$$\delta\left\{V\rho^{\Psi}\left(\phi_i, \vec{r}_j, \zeta_k^\star(\phi_i, \vec{r}_j)\right)\right\}\big|_{\delta\phi=0} = V\vec{\nabla}\vec{u} : \sigma \tag{10}$$

where σ is by definition the *Maxwell stress tensor* and $\vec{\nabla}\vec{u}(\vec{r}_j, \delta\vec{r}_j)$ is the gradient of the displacement field obtained by perturbing the $\vec{r}_j$ vectors (virtual displacement). The total variation of energy is

$$\delta\left\{V\rho^{\Psi}\right\} = \delta V\ \rho^{\Psi} + V\frac{\partial\rho^{\Psi}}{\partial\phi_i}\delta\phi_i + V\frac{\partial\rho^{\Psi}}{\partial\vec{r}_j}\delta\vec{r}_j + V\frac{\partial\rho^{\Psi}}{\partial\zeta^k}\delta\zeta_k. \tag{11}$$

The second term at the r.h.s. plays no role for forces because $\delta\phi = 0$ in (10). The fourth term does not play a role either, because (9) implies that either $\frac{\partial\rho^{\Psi}}{\partial\zeta^k} = 0$ or $\delta\zeta_k = 0$ at equilibrium. The only thing that remains to do now, is to factorize the two remaining terms into the form of the r.h.s. of (10). This will be done in the full paper.

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