Error Estimators for Proper Generalized Decomposition in Time-Dependent Electromagnetic Field Problems

F. Müller[1], T. Henneron[2], S. Clénet[3], and K. Hameyer[1]

[1]Institute of Electrical Machines (IEM), RWTH Aachen University, 52062 Aachen, Germany.
[2]L2EP, Université de Lille, 59655 Lille, France.
[3]L2EP, Ecole Nationale Supérieure d'Arts et Métiers, 59046 Lille, France.

Due to fine discretization in space and time, the simulation of transient electromagnetic phenomena results in a large system of equations. To cope with this computational effort, model-order reduction techniques can be employed. To assess the accuracy of the solution of the reduced model, an error estimation is crucial. A commonly used approach consists in the evaluation of the deviation between the reduced and the full model. This yields a loss of the *a priori* property of the proper generalized decomposition. To overcome this problem, two *a priori* criteria are presented in this article.

***Index Terms*— Error criteria, finite-element method (FEM), model-order reduction (MOR), proper generalized decomposition (PGD).**

I. Introduction

LARGE-SCALE finite-element models arise from, e.g., time-dependent electromagnetic-field problems, due to the skin depths of the eddy currents. On the one hand, to properly model eddy currents, the conducting regions have to be accurately discretized in space. On the other hand, the time interval has to be sampled accurately to consider all transient effects. The resulting computational effort of these transient simulations can be reduced by model-order reduction (MOR) techniques. The reduction techniques can be distinguished in two classes, namely, *a posteriori* and *a priori* methods. One of the well-known *a posteriori* methods is the proper orthogonalized decomposition (POD), which is based on collecting snapshots of the reference system to calculate a reduced representation. *A priori* methods such as the proper generalized decomposition (PGD) method construct a reduced-order model (ROM) without any previously obtained solutions [1]. The PGD has been applied to different problems in mechanics [1]–[4] and electromagnetics [5]–[10], [12], [13] and represents a desirable strategy to solve engineering problems. While different error criteria for *a posteriori* methods have been formulated [6], a reasonable criterion for *a priori* methods has not been stated yet. To maintain the *a priori* property of the PGD, an *a priori* error criterion is presented in the following.

II. Magnetoquasi-static Problem

To solve the magnetoquasi-static field problem, the finite-element method (FEM) with the magnetic vector potential $\boldsymbol{A}$ is employed (1). The problem consists of a domain with unary boundary conditions and a conducting subdomain which allows eddy currents

$$\nabla \times \nu(\nabla \times \boldsymbol{A}(t)) + \frac{\sigma \partial \boldsymbol{A}(t)}{\partial t} = \boldsymbol{J}(t). \quad (1)$$

III. Proper Generalized Decomposition

A. Basic Approach

The basic principle of the PGD is to decompose the solution of a linear partial differential equation (PDE) into a sum of m tensor products [1], [2], [9]

$$\boldsymbol{A}(\mathrm{x}, \mathrm{t}) \approx \sum_{\mathrm{i}=1}^{\mathrm{m}} \boldsymbol{R}_{\mathrm{i}}(x) S_{\mathrm{i}}(t). \quad (2)$$

$\boldsymbol{R}(x)$ is the space-related part of the solution, while the $S(t)$ contains the time dependence. The number of terms m in the expansion (2) is called the number of modes. An alternative direction scheme is adapted to enrich the PGD basis [5].

B. Exploiting Superposition Principle

For linear problems with several excitation sources, such as multiple coils, the general approach stated above may not converge or produce adverse results. Due to the linear nature of the field problem in the absence of nonlinear materials, the superposition of the fields produced by each of the k sources adds up to the total field distribution, and therefore, an adapted approach is employed [12]

$$\boldsymbol{A}(\mathrm{x}, \mathrm{t}) \approx \sum_{\mathrm{j}=1}^{\mathrm{k}} \sum_{\mathrm{i}=1}^{\mathrm{m}} \boldsymbol{R}_{\mathrm{j,i}}(x) S_{\mathrm{j,i}}(t). \quad (3)$$

Substituting (3) into (1) leads to a differential algebraic equation for computing the space mode (4) and an ordinary

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differential equation (5) for the time mode

$$\sum_{\mathrm{j}=1}^{\mathrm{k}}\sum_{\mathrm{i}=1}^{\mathrm{m}}\int_{\mathrm{t}} S_{\mathrm{j,i}}(t)S_{\mathrm{j,m}}(t)dt\int_{\mathrm{G}}\nu\nabla\times R_{\mathrm{j,i}}\nabla\times R_{\mathrm{j,m}}^{*}dG + \sum_{\mathrm{j}=1}^{\mathrm{k}}\sum_{\mathrm{i}=1}^{\mathrm{m}}\int_{\mathrm{t}}\frac{\partial S_{\mathrm{j,i}}(t)}{\partial t}S_{\mathrm{j,i}}(t)dt\int_{\mathrm{G}}\sigma R_{\mathrm{j,i}}R_{\mathrm{j,m}}^{*}dG = \sum_{\mathrm{j}=1}^{\mathrm{k}}\int_{\mathrm{t}} I_{\mathrm{j}}(t)S_{\mathrm{j,m}}(t)dt\int_{\mathrm{G}} N_{\mathrm{j}}(x)R_{\mathrm{j,m}}^{*}dG \quad (4)$$

$$\sum_{\mathrm{j}=1}^{\mathrm{k}}\sum_{\mathrm{i}=1}^{\mathrm{m}} S_{\mathrm{j,i}}(t)\int_{\mathrm{G}}\nu\nabla\times R_{\mathrm{j,i}}\nabla\times R_{\mathrm{j,m}}dG + \sum_{\mathrm{j}=1}^{\mathrm{k}}\sum_{\mathrm{i}=1}^{\mathrm{m}}\frac{\partial S_{\mathrm{j,i}}(t)}{\partial t}\int_{\mathrm{G}}\sigma R_{\mathrm{j,i}}R_{\mathrm{j,m}}dG = \sum_{\mathrm{j}=1}^{\mathrm{k}}\int_{\mathrm{t}} I_{\mathrm{j}}(t)S_{\mathrm{j,m}}(t)dt\int_{\mathrm{G}} N_{\mathrm{j}}(x)R_{\mathrm{j,m}}dG. \quad (5)$$

This leads to the solution of k meta models which need to be analyzed in terms of convergence of the metamodel mode enrichment process as well as the absolute convergence of the full decomposition. The principle is depicted in Fig. 1, which is followed until every submodel $j = [1, k]$ has converged. Due to superposition, the submodels do not depend on each other and can therefore be computed with a fixed j and superposed in the post-processing stage. For stabilizing the enrichment procedure, one of the two entities $\boldsymbol{R}$ or S should be normed to prevent divergence, if one entity tends to converge toward zero while the other diverges to infinity. In our case, we normed the space modes.

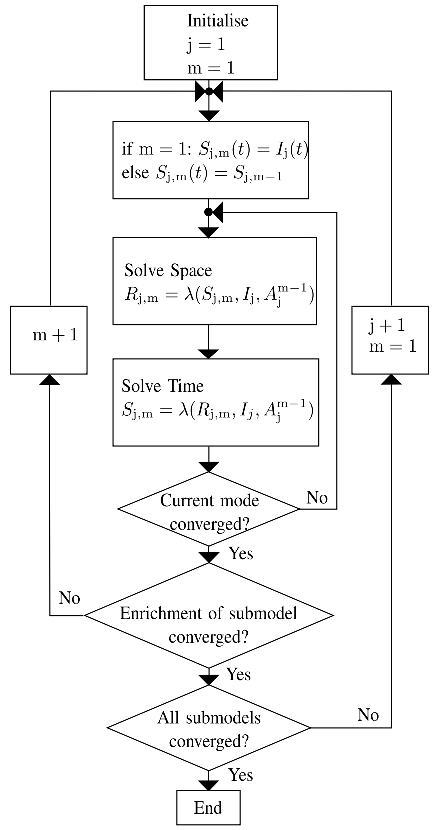

Fig. 1. Alternating scheme to determine linear PGD representation.

IV. Accuracy of the PGD

Even though the PGD is applied to many areas, the error evaluation and the information content of the single modes were not the main focus of research. The enrichment is terminated after a certain *a posteriori* relative error is fulfilled or until a defined number of modes are enriched [5], [7]–[9]. To overcome this disadvantage, different error criteria are introduced and compared in this article.

A. A Posteriori Error Criteria

The need for a reference solution, which has to be obtained from the complete system of equations, characterizes *a posteriori* error criteria. Common criteria in this context use the magnetic energy, the Joule losses (6), or the reference solution $\mathbf{X}_{\mathrm{ref}}$ (7) evaluated using the two norms

$$\epsilon_j = \frac{||P_{j,\mathrm{ref}} - P_{\mathrm{j,PGD}}||_2}{||P_{\mathrm{j,ref}}||_2} \quad (6)$$

$$\epsilon_{\mathrm{ref}} = \frac{||X_{\mathrm{PGD}} - X_{\mathrm{ref}}||_2}{||X_{\mathrm{ref}}||_2}. \quad (7)$$

B. A Priori Error Criteria

One option to evaluate the relative convergence of the enrichment process is comparing the norms of the difference between the PGD model with m and $m-1$ modes (8) [5]–[10]. The mentioned criterion is a measure for the relative convergence of the decomposition process, but is missing to provide information about the absolute error

$$\epsilon_{\mathrm{sol}} = \frac{||X_{\mathrm{PGD}}^{m} - X_{\mathrm{PGD}}^{m-1}||_2}{||X_{\mathrm{PGD}}^{m-1}||_2}. \quad (8)$$

To retain the *a priori* property of the PGD and to cope with disadvantages of [8], two criteria are presented in the following paragraphs. Combining these two leads to a reasonable measure of relative and absolute convergence of the decomposition.

1) Absolute Residual: Instead of comparing the reference solution to the PGD solution, it is more convenient to compute the absolute residual (6). Although a reference solution is not required, the evaluation of all time steps in (9), with the reference system matrix $\mathbf{M}$ and the time-dependent excitation $\boldsymbol{B}_{\mathrm{PGD}}(t)$, is still necessary, resulting in high computational efforts. This criterion can be interpreted as an *a priori* version of (7) and yields the absolute residual

$$\epsilon_{\mathrm{Abs}} = \frac{||\mathbf{M}\boldsymbol{X}_{\mathrm{PGD}}(t) + \boldsymbol{B}_{\mathrm{PGD}}(t)||_2}{||\boldsymbol{B}_{\mathrm{PGD}}(t)||_2} \quad (9)$$

$$\boldsymbol{B}_{\mathrm{PGD}} = \boldsymbol{J}(t) - \sigma\frac{\boldsymbol{A}_{\mathrm{PGD}}(t-1)}{dt}. \quad (10)$$

2) Information Content: An additional approach can be formulated by using the singular value decomposition (SVD). Under the assumption that the singular values of the system decrease rapidly, they can be used as a measure of

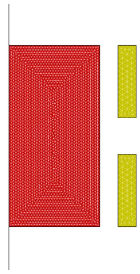

Fig. 2. Transient academic example (**Multi Coil Model**): magnetizing coils (green) and conductive sample (red).

convergence of the enrichment. The evaluation of the PGD solution in a certain time step can be reformulated into matrix form by

$$\boldsymbol{A}(x,t) \approx \sum_{\mathrm{j}}^{\mathrm{k}} \sum_{i=1}^{m} \boldsymbol{R}_{\mathrm{j,i}}(x) S_{\mathrm{j,i}}(t) = \sum_{\mathrm{j}}^{\mathrm{k}} \mathbf{M}_{\mathrm{R,j}} \cdot \boldsymbol{S}_{\mathrm{j}}. \quad (11)$$

In (11), $\mathbf{M}_{\mathrm{R}}$ is a matrix with the space modes, $\boldsymbol{R_i}$ as columns, and $\boldsymbol{S}$ is a vector with the values of $S_i(t)$ in the evaluation timestep as entries. The matrix $\mathbf{M}_{\mathrm{R}}$ can be decomposed by a SVD and the resulting singular values give a hint for the information content of the modes, since $\mathbf{M}_{\mathrm{R}}$ acts as a linear projection on $\boldsymbol{S}$. In order to cope with the high computational effort of the SVD, one property of the singular values can be considered as an advantage. In order to avoid the high effort of SVD, the eigen decomposition of the matrix $\mathbf{M}_{\mathrm{R}}^{\mathrm{T}}\mathbf{M}_{\mathrm{R}}$ can be used.

V. SIMULATION

The above discussed criteria are applied to a Cartesian academic example holding linear material properties. The model consists of a conductive sample in combination with two excitation coils (see Fig. 2). The conductivity of the sample is equal to 10 kS/m, while the reluctivity of the sample is an arbitrary set to 2183.2 Am/Vs. The absolute error tolerance was set to 0.3 per mil. The domain is bound by a Dirichlet boundary condition in a reasonable distance to the area of interest and at the y-axis. The basic approach (2) is not able to model the transient field distribution for multiple coils, especially if they have a phase lag. Therefore, this model is operated with one sinusoidal-fed coil and one cosinusoidal fed coil. The cosinussoidal signal is ramped up in the first fourth of a period. Using the adapted PGD approach (3) leads to a reasonable decomposition. The tolerance regarding the absolute residual for the multicoil model is set to 0.3 per mil. From Fig. 3, it can be depicted that the eddy current losses of the ROM are in good agreement with the reference solution. The error of the Joule losses is depicted in Fig. 4. The multicoil model needs at least 33 modes to achieve an error 6 smaller than 1%. Fig. 5 shows the singular values of $\mathbf{M}_{\mathrm{R}}$ (11), and it can be recognized that the singular values decrease rapidly after a certain number of modes is enriched. The absolute

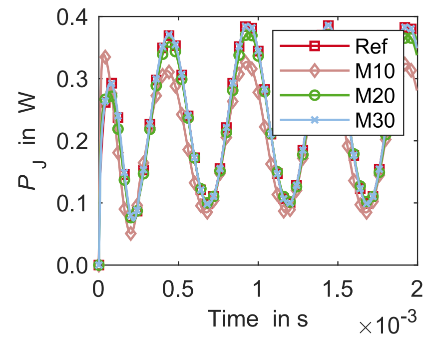

Fig. 3. Eddy current losses in the conductive sample.

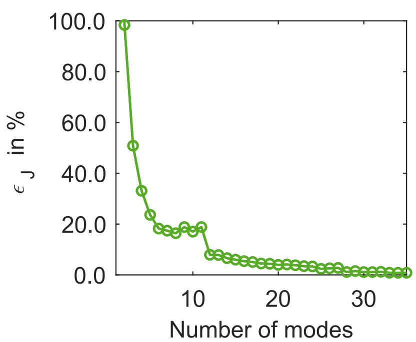

Fig. 4. Relative error of the eddy current losses.

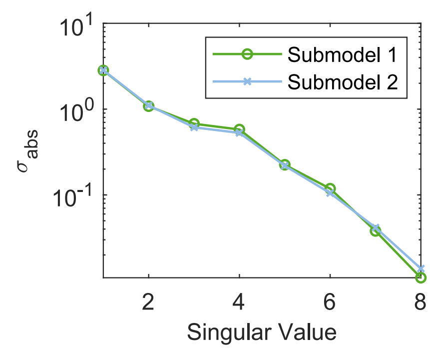

Fig. 5. Singular values of $\mathbf{M}_{\mathrm{R}}$.

error is depicted in Fig. 6. Comparing the absolute error in Fig. 6 with Fig. 5 shows that the singular values indicate a convergence of the enrichment process, but is too coarse to be sufficient on its own, especially for complex models. The absolute mathematical error (9) holds a reasonable error indicator, due to the fact that it analyzes the accuracy of the reduced solution in the context of the reference system. To evaluate the mathematical error, the residual has to be built in each time instance and leads to the additional computational effort. To diminish this effort, the mathematical error is first evaluated after the singular values significantly decreased ($\sigma_{\max}/\sigma_{\min} = 0.01$) in comparison to the first. In Fig. 6, it is obvious that the decomposed model is not optimal and converges very slowly. To improve the convergence behavior, the decomposition can be further improved by using an update step of all time functions, depicted by **Multi-U** in Fig. 6 [1]. Due to the fact that the MOR extracts most of the relevant

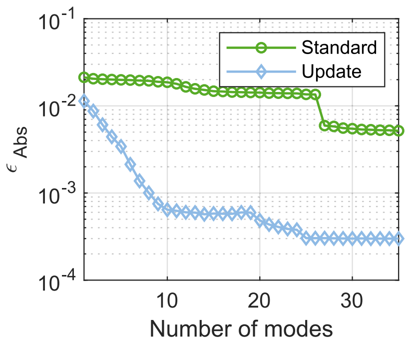

Fig. 6. Absolute residual error.

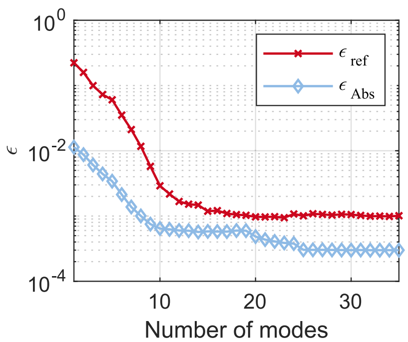

Fig. 7. Comparison of *a priori* (ϵ_{Abs}) and *a posteriori* (ϵ_{ref}) residual error.

TABLE I
TIME OF ERROR CRITERIA EVALUATION

PGD		A-Posteriori	A-Priori	
Offline	**Online**	**Joule Losses**	***SVD***	***Math. Res.***
82 s	2 s	21 s	0.1 s	52 s

information, there is still a certain loss in accuracy compared to the reference in terms of the absolute residual. Furthermore, the PGD does not necessarily have to be orthogonal, in contrast to the POD [1]. Even though the not updated multicoil model produces accurate Joule losses, the absolute residual is quite high. This clearly indicates the difference between comparing Joule losses (6) and the mathematical residual (9). While (6) is a globally integrated quantity, (9) is the true residual of the system. In addition, it is obvious that the decomposition is crucially improved by the update. Finally, the errors computed by (7) and (9) are compared here, because (9) can be interpreted as an a-prior version of (7). Both errors are depicted in Fig. 7 and it can be seen that the overall behavior is similar but a certain offset is visible. This deviation originates from the load vector (10), which depends on the decomposition. This leads to the conclusion that (9) can be used as a convergence indicator. The combination of both *a priori* criteria are competitive to the *a posteriori* criteria in terms of accuracy, but do not need reference solutions and hold less computational effort. The time for the model creation and different error criteria is given in Table I, related to the multicoil model. The reference needs 232 s for the multicoil model, while the PGD enrichment needs 82 s, excluding the calculation of (9).

VI. CONCLUSION

A measure for convergence of the enrichment process of the PGD is proposed, which is not based on reference solutions. In combination with (9), the *a priori* property of the PGD can be kept, while receiving important information about the relative and absolute convergence. Consecutively, the comparison between commonly used error and convergence indicators points the advantages of the proposed *a priori* criteria out.

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