Observations about conserved quantities in electromagnetism and about the computation of forces

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Overview

- **2-forms** and **flux tubes** in space $\mathbb{S}$
- **3-forms** and **spacetime tubes** in spacetime $\mathbb{S} \times \mathbb{T}$
- **solenoidal** quantities vs **conserved** quantities
- intrinsic speed of conserved quantities in $\mathbb{S} \times \mathbb{T}$
- application to a plane wave

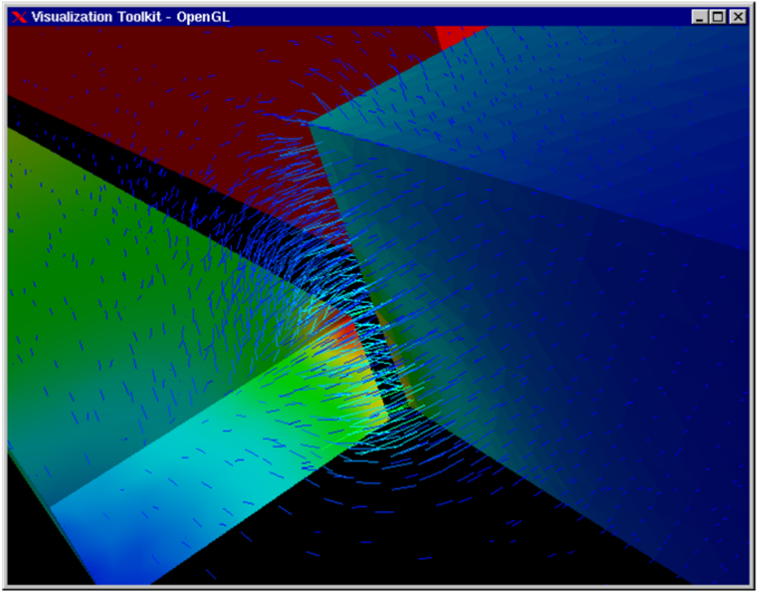
Equivalence 2-form /vector field in $\mathbb{S}$

- 3D space $\mathbb{S}$
- volume form : $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$
- equivalence between a 2-form b (2=3-1) and a vector field **v**

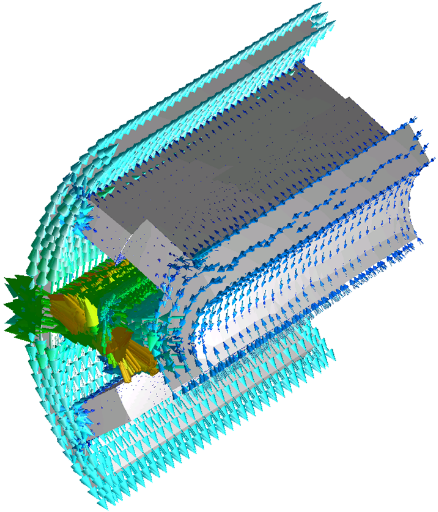
$$\mathrm{i}_{\mathbf{v}}\omega = b \quad \Rightarrow \quad \mathrm{i}_{\mathbf{v}} b = 0$$

- the congruency associated with **v** is a set of curves in $\mathbb{S}$
- notion of **flux line**
- Take the vector **v** at any point P, take **any other** vector at that point, they form a facet not crossed by any flux.

Flux lines

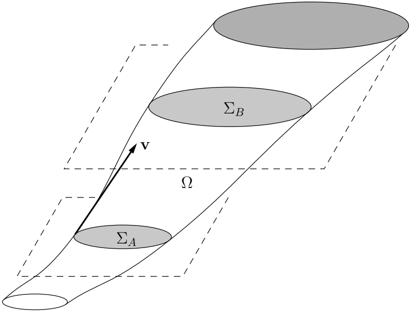


Flux lines



Problem inherent to the “vector” representation of 2-forms

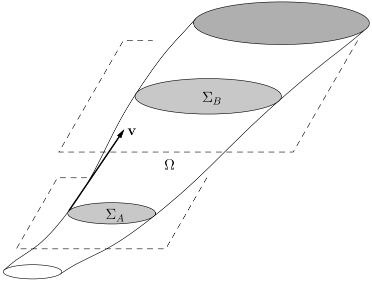
Flux tube



A **Flux tube**

- generated by **Lie dragging** (with $\mathbf{v}(b)$) the boundary $\partial\Sigma$ of any surface Σ.
- has a **tangent space** that always contain **v** such that $\mathrm{i}_{\mathbf{v}} b = 0$.
- is never crossed by any flux, irrespective of b being solenoidal or not.

Solenoidality



- b solenoidal $\equiv$ b closed : $\mathrm{d}b = 0$
- b non solenoidal : $\mathrm{d}b = \rho$
- Stokes : $\int_\Omega \mathrm{d}b = \int_{\partial\Omega} b$
- $\partial\Omega = \Sigma_A - \Sigma_B + FT$
- $\phi_A - \phi_B = \int_\Omega \rho$ (*)
- if $\rho = 0$, $\phi_A = \phi_B$

- Solenoidality implies one has the same flux when sectioning the flux tube at different position, $\phi_A = \phi_B$.
- For non-solenoidal fields, the flux may be different, but exclusively by the effect of the sources ρ in Ω (*) . . .
- . . . and **not** because of some flux crossing the flux tube.

Equivalence 3-form /vector field in $\mathbb{S} \times \mathbb{T}$

- (3+1)D spacetime, $\mathbb{S} \times \mathbb{T}$ (product space)
- volume form : $\omega = \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \wedge \mathrm{d}t$
- equivalence between a 3-form E (3=4-1) and a vector field **V**

$$\mathrm{i}_{\mathbf{V}}\omega = E \quad \Rightarrow \quad \mathrm{i}_{\mathbf{V}}E = 0$$

- the congruency associated with **V** is now a set of spacetime trajectories
- notion of **worldline**

Space vs time splitting / Conservation equation

- Before generalizing the notion of flux tube, a word about the spacetime structure of 3-forms.
- Any 3-form E in $\mathbb{S} \times \mathbb{T}$ can be written

$$E = \rho - S \wedge \mathrm{d}t$$

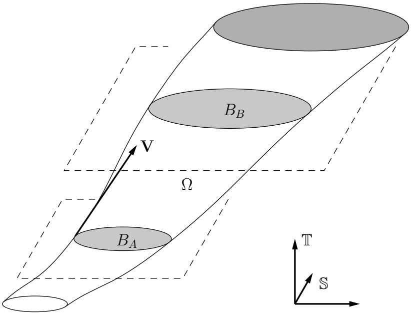
in terms of a 3-form ρ and a 2-form S, both defined on $\mathbb{S}$.

- **Conservation equation** : if E is closed,

$$\mathrm{d}E = 0 \quad \Leftrightarrow \quad \partial_t \rho + \mathrm{d}_u S = 0$$

- **Conserved quantity** in $\mathbb{S} \times \mathbb{T} \leftrightarrow$ **solenoidal quantity** in $\mathbb{S}$
- Neither ρ nor S is a conserved quantity on its own.
- Only the compound spacetime 3-form E is a conserved quantity.
- E.g. system with only positive charges :
  ρ is the charge density and S is the associated current density.

Spacetime tubes

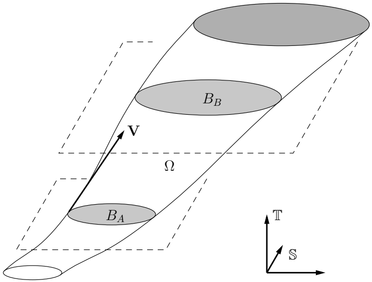


A **spacetime tube** *(same picture to highlight analogy)*

- is generated by **Lie dragging** (with now $\mathbf{V}(E)$) the boundary ∂B of any box $B \subset \mathbb{S}$.
- has a **tangent space** that always contain $\mathbf{V}(E)$.
- is never crossed by E irrespective of E being conservative or not.

Nota : In spacetime, the integration of the 3-form E over a 3-dimensional submanifold has the meaning of a flux crossing the submanifold.

Spacetime tubes (contd)



- E conserved : $\mathrm{d}E = 0$
- E not conserved : $\mathrm{d}E = \dot{Q}$
- Stokes : $\int_\Omega \mathrm{d}E = \int_{\partial\Omega} E$
- $\partial\Omega = B_A - B_B + ST$
- $\Psi_A \equiv \int_{B_A} E$, $\Psi_B \equiv \int_{B_B} E$
- $\Psi_A - \Psi_B = \int_\Omega \dot{Q}$
- if $\dot{Q} = 0$, $\Psi_A = \Psi_B$

- Each spacelike section of the spacetime tube, at time t, is the boundary of a box $B \subset \mathbb{S}$: B_A and B_B at t_A and t_B resp.
- If $\mathrm{d}E = 0$, the boxes B_A and B_B contain **the same amount of the conserved quantity**, $\Psi_A = \Psi_B$ (same charges, same photons).
- If $\mathrm{d}E = \dot{Q}$, the enclosed quantities Ψ_A and Ψ_B may be different, but exclusively by the effect of the sources $\dot{Q}$ in Ω.
- . . . and **not** because of some leakage of E through the spacetime tube.

Spacetime tubes contain velocity information

- ...but remember the vector field **V** is not an external flow.
- It is just another way of looking at E.
- A closed 3-form in spacetime contains information about the motion (and hence the velocity) of boxes that enclosed fixed amounts of the conserved quantity E.
- If

$$E = \rho - S \wedge \mathrm{d}t,$$

this (space) velocity **v** is the one given by

$$\mathrm{i}_{\mathbf{v}} \rho = S.$$

Examples

$$\mathbf{S} = \rho \mathbf{v}$$

ρ	S
charge density	current density
mass density	momentum density
electromagnetic energy density	Poynting vector

BUT one has

- to consider S and ρ as fundamental quantities, and $\mathbf{v}$ as a derived quantity,
- to refrain considering S and ρ as independent fields.
- In particular, **electromagnetic energy density** and the **Poynting vector** are the two components of one single conserved quantity :

$$E = \rho - S \wedge \mathrm{d}t$$

Application to a plane wave in empty space

Let $M = \mathbb{S} \times \mathbb{T}$, with coordinates $\{u, v, w, \tau\}$, and the plane wave

$$e = E_0 \cos(kw - \omega\tau)\,\mathrm{d}u$$

Calculating

$$a = -\int e\,\mathrm{d}\tau \to b = \mathrm{curl}\,a \to h = *\mu_0^{-1} b \to d = -\int \mathrm{curl}\,h\,\mathrm{d}\tau \to e = *\varepsilon_0^{-1} d$$

one obtains by identification

$$\varepsilon_0 \mu_0 \frac{\omega^2}{k^2} = 1$$

and one defines the **phase velocity** of the wave

$$c = \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}.$$

Plane wave in M

One calculates also the **energy density** :

$$\rho = \varepsilon_0 E_0^2 \cos^2(kw - \omega\tau)\ \mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}w$$

and the **Poynting vector** :

$$S \equiv e \wedge h = c\varepsilon_0 E_0^2 \cos^2(kw - \omega\tau)\ \mathrm{d}u \wedge \mathrm{d}v$$

showing that

$$S = \mathrm{i}_{\mathbf{v}}\rho$$

with

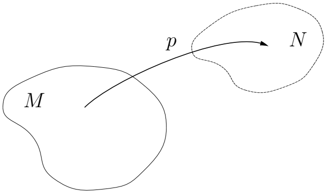
$$\mathbf{v} = c\partial_w$$

the wave's velocity.

Galilean transformation : $p : M \mapsto N$

$$p : M \mapsto N, p\{u, v, w, \tau\} = \{x, y, z + Wt, t\}$$

$$\begin{aligned} p^{-*}\,\mathrm{d}u &= \mathrm{d}x \\ p^{-*}\,\mathrm{d}v &= \mathrm{d}y \\ p^{-*}\,\mathrm{d}w &= \mathrm{d}z - W\,\mathrm{d}t \\ p^{-*}\,\mathrm{d}\tau &= \mathrm{d}t \end{aligned}$$



$$p^{-*}e = E_0 \cos(k(z + Wt) - \omega t)\,\mathrm{d}x = E_0 \cos(kz - (\omega - kW)t)\,\mathrm{d}x$$

according to which the new **phase velocity** of the wave is

$$\frac{\omega - kW}{k} = c - W.$$

Galilean transformation : $p : M \mapsto N$

Transformed **energy density** :

$$
\begin{aligned}
p^{-*}\rho_M =& \varepsilon_0 E_0^2 \cos^2(kz - (\omega - kW)t)\ \mathrm{d}x \wedge \mathrm{d}y \wedge (\mathrm{d}z - W\,\mathrm{d}t) \\
=& \varepsilon_0 E_0^2 \cos^2(kz - (\omega - kW)t)\,\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
& - \left[W\varepsilon_0 E_0^2 \cos^2(kz - (\omega - kW)t)\ \mathrm{d}x \wedge \mathrm{d}y \right] \wedge \mathrm{d}t
\end{aligned}
$$

Transformed **Poynting vector** :

$$
p^{-*}S_M = (c - W)\varepsilon_0 E_0^2 \cos^2(kz - (\omega - kW)t)\ \mathrm{d}x \wedge \mathrm{d}y
$$

- The transformed energy density contains Poynting vector part
- ... that cancels the second term in $p^{-*}S_M$.

One has thus : $p^{-*}(\rho_M - S_M \wedge \mathrm{d}\tau) = \rho_N - S_N \wedge \mathrm{d}t$

$$
\begin{aligned}
\rho_N =& \varepsilon_0 E_0^2 \cos^2(kz - (\omega - kW)t)\ \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\
S_N =& c\varepsilon_0 E_0^2 \cos^2(kz - (\omega - kW)t)\ \mathrm{d}x \wedge \mathrm{d}y
\end{aligned}
$$

Invariance of velocity

One has finally

$$\begin{aligned}\rho_N =& \varepsilon_0 E_0^2 \cos^2(kz - (\omega - kW)t)\ \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \\ S_N =& c\varepsilon_0 E_0^2 \cos^2(kz - (\omega - kW)t)\ \mathrm{d}x \wedge \mathrm{d}y\end{aligned}$$

so that

$$S_N = \mathrm{i}_{\mathbf{v}_N} \rho_N$$

with

$$\mathbf{v}_N = c\partial_z$$

exactly like one had

$$S_M = \mathrm{i}_{\mathbf{v}_M} \rho_M$$

with

$$\mathbf{v}_N = c\partial_w$$

Conclusion

- Energy density, as any complete 3-form in spacetime, contains information about the velocity at which energy propagates.
- This velocity is invariant, contrary to the phase velocity, which is not.
- A **diffeomorphism invariant formulation** of electromagnetism (used in the derivation above) follows from the definition of two operators :

  $$\mathcal{L}_p = p^{-*}\partial_\tau p^* \quad , \quad \mathrm{d}_p = p^{-*}\,\mathrm{d}u^l \wedge \partial_{u^l} p^*$$

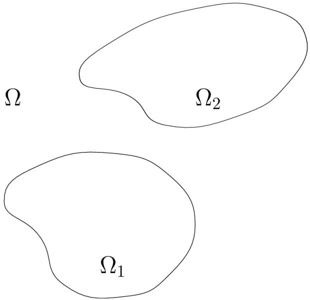
  that are the mapping of the **partial time derivative** ∂_τ and of the **space exterior derivative** $\mathrm{d}u^l \wedge \partial_{u^l}$.
- It is formulation in the spirit of Hertz.
- It contains the **Lorentz invariant formulation** as a particular case.
- In particular, $\mathcal{L}_p$ is a kind of **Lie derivative**.
- It allows working with **non-inertial frames** in electromagnetism, which makes **electromechanical coupling** (nearly) trivial, by factorizing $\mathrm{grad}\,\mathbf{v}$. Applications : Maxwell stress tensor, electromagnetic forces, upwind operator, . . .

Air gap power, Poynting vector

Assuming an absolute time

$$\mathcal{L}_p = \partial_t + \mathcal{L}_{\mathbf{W}}$$

$$\begin{aligned}
\int_\Omega \mathcal{L}_{\mathbf{W}}\, \rho - \int_{\partial\Omega} S_{conv} &= \\
\int_\Omega \mathcal{L}_{\mathbf{W}} \frac{\mathbf{b}\cdot\mathbf{h}}{2} - \int_{\partial\Omega} \mathbf{h} \times \mathcal{L}_{\mathbf{W}}\, \mathbf{a} &= \\
\int_\Omega \mathrm{div}\,(\frac{\mathbf{b}\cdot\mathbf{h}}{2}\mathbf{W}) - \int_{\partial\Omega} \mathbf{h} \times (\mathrm{grad}\,(\mathbf{a}\cdot\mathbf{W}) - \mathbf{W}\times\mathbf{b}) &= \\
\int_{\partial\Omega} \{\frac{\mathbf{b}\cdot\mathbf{h}}{2}\mathbf{W} - \mathbf{b}\cdot\mathbf{h}\,\mathbf{W} + \mathbf{b}\,\mathbf{h}\cdot\mathbf{W}\} &= \\
\int_{\partial\Omega} \left(\mathbf{b}\,\mathbf{h} - \frac{\mathbf{b}\cdot\mathbf{h}}{2}\mathbb{I}\right) &: \mathbf{W}
\end{aligned}$$



- In empty space, the **Lie derivative of the magnetic energy** and the **convective part of the Poynting vector** form together the mechanical work delivered to solid objects by the **Maxwell stress tensor**.

Co-moving time (or convective) derivative D_t

Interpretation

$$p_t^{-*} \partial_t p_t^* = D_t \quad , \quad \partial_t \int_\Omega \alpha = \int_\Omega D_t \alpha$$

Lie derivative

$$D_t \mathbf{a} = \partial_t \mathbf{a} + \mathcal{L}_{\mathbf{v}}\, \mathbf{a}$$

Cartan's magic formula

$$\mathcal{L}_{\mathbf{v}}\, \mathbf{a} = \mathrm{d} \mathrm{i}_{\mathbf{v}} \mathbf{a} + \mathrm{i}_{\mathbf{v}}\, \mathrm{d}\mathbf{a}$$

Expression for differential forms of degree 0-3 :

$$\begin{aligned}
\mathcal{L}_{\mathbf{v}}\, f &= \mathbf{v} \cdot (\mathrm{grad}\, f) \\
\mathcal{L}_{\mathbf{v}}\, \mathbf{a} &= \mathrm{grad}\, (\mathbf{a} \cdot \mathbf{v}) - \mathbf{v} \times \mathrm{curl}\, \mathbf{a} \\
\mathcal{L}_{\mathbf{v}}\, \mathbf{b} &= \mathrm{curl}\, (\mathbf{b} \times \mathbf{v}) + \mathbf{v}\, \mathrm{div}\, \mathbf{b} \\
\mathcal{L}_{\mathbf{v}}\, \rho &= \mathrm{div}\, (\rho \mathbf{v})
\end{aligned}$$