Field and field-circuit models of electrical machines

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Abstract
Purpose – The purpose of this paper is to develop and systemize the 3D finite element (FE) description of electromagnetic field in electrical machines.

Design/methodology/approach – 3D FE models of electrical machines are considered. The model consists of FE equations for the magnetic field, equations describing eddy currents and equations, which describe the currents in the machine windings. The FE equations are further coupled by the electromagnetic torque to the differential equation of motion. In the presented field-circuit model, the flux linkages with the windings are expressed by two components. Attention is paid to the description of machine winding. Both scalar and vector potential formulations are analysed. The FE equations are derived by using the notation of circuit theory. The methods of movement simulation and torque calculation in FE models are discussed.

Findings – Proposed circuit description of electromagnetic field in electrical machines conforms to the applied method of electric and magnetic circuit analysis. The advantage of the presented description is that the equations of field model can be easy associated with the other equations of the electric drive system.

Originality/value – The applied analogies between the FE formulation and the equivalent magnetic and electric network models help formulate efficient field models of electrical machines. The developed models after coupling to the models of supply and control system can be successfully used in the analysis and design electric drives.

Keywords Electric machines, Finite element analysis, Magnetic fields, Eddy currents, Modelling

Paper type Research paper

I. Introduction
In the paper, we systemize the applied models of electrical machines and actuators. We consider machine models describing the electromagnetic phenomena and characteristics for steady state and transients operation.

The models, described by the systems of ordinary differential equations with inductances, will be considered as the circuit models. In the field models winding inductances do not exist. Flux linkages and electromagnetic torques and forces are calculated using field quantities. The field equations are coupled to the equations describing the winding connections and contain terms defined by field quantities and lumped parameters (Arkkio, 1988; Demenko, 1994; Piriou and Razek, 1993; Tsukerman *et al.*, 1993) The models of this type can be considered as field-circuit couplings (De Gersem *et al.*, 1998, 2000; Lahaye *et al.*, 2002; Sadowski *et al.*, 1993; Strangas, 1985).

In the paper, particular attention is paid to field and field-circuit coupled models of typical electrical machines and actuators.

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II. Finite element (FE) equations of magnetic field

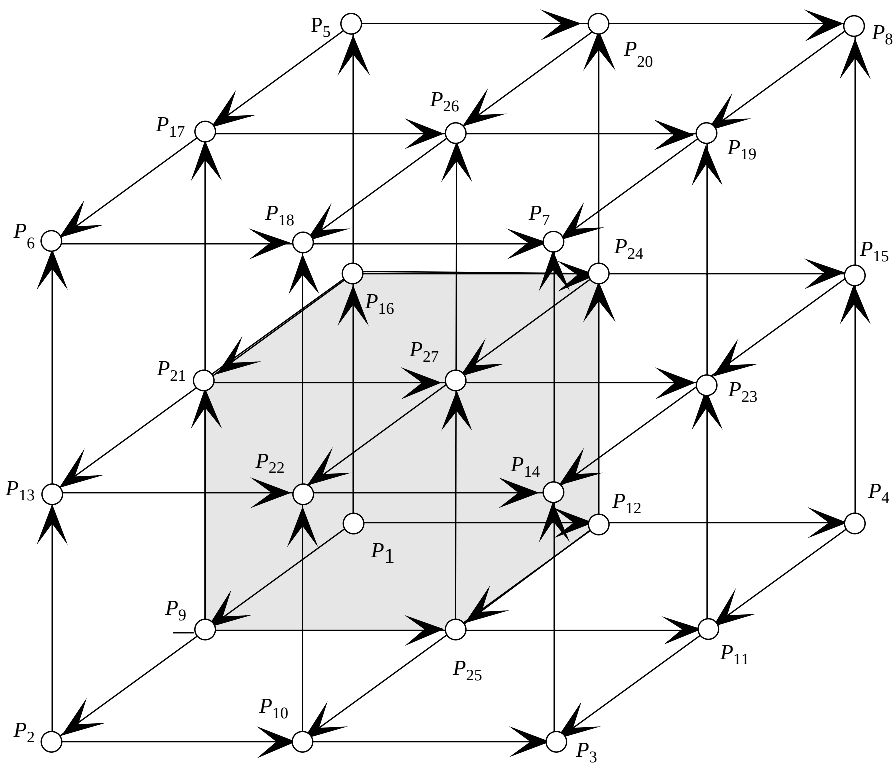
Two most popular FE formulations are discussed:

(1) a formulation using scalar potential Ω and nodal elements; and

(2) a formulation using vector potential and edge elements.

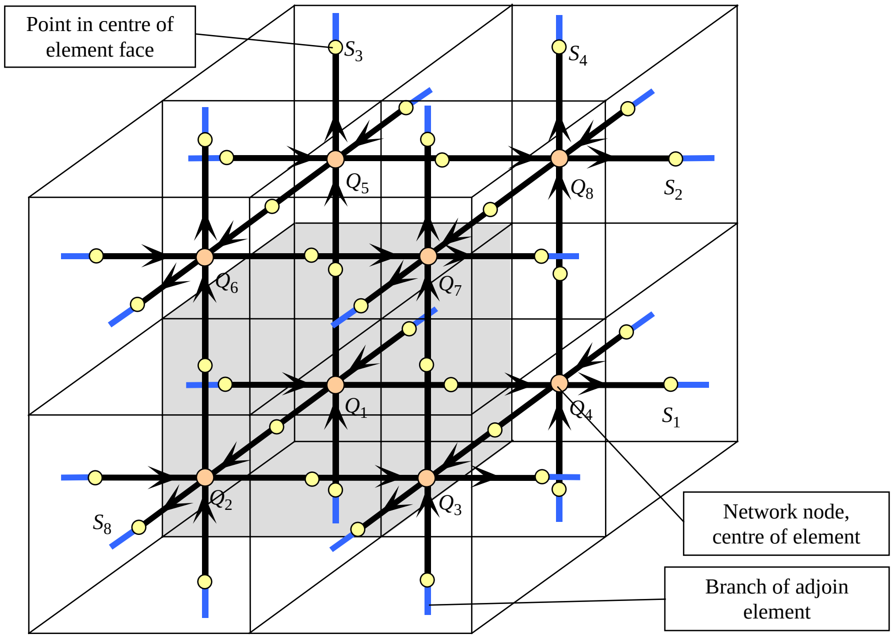
In formulation (1), polynomial interpolating functions $\Omega(x,y,z)$ are constructed on the nodal values of Ω, i.e. on the nodal potential. Formulation (2) applies edge value of $\boldsymbol{A}$. For an oriented edge P_1P_2, the edge value of $\boldsymbol{A}$ is equal to the line integral of $\boldsymbol{A}$ on P_1P_2 (Demenko and Sykulski, 2006). The edge value of $\boldsymbol{A}$ for edge P_1P_2 can be considered as a loop flux in the loop around P_1P_2.

The papers (Demenko and Sykulski, 2002; Demenko and Sykulski, 2006) report that FE equations represent nodal and loop equations of two types of networks: "edge networks" (EN) where branches are associated with the edges of the elements, and "facet networks" (FN) with branches connecting the centres of the relevant facets with the centre of the element volume. The FE model composed of eight-node hexahedrons is used to illustrate these networks (Figures 1 and 2). The structural matrices of the networks are the FE representations of differential operators. For example, the transposed nodal incidence matrix $\boldsymbol{k}_n$ of EN represent the "grad" operator and the transposed loop matrix $\boldsymbol{k}_e$ for FN is the network representation of the



Source: Demenko and Sykulski (2006)

Figure 1.
Edge graph of eight hexahedrons



Source: Demenko and Sykulski (2006)

Figure 2.
Facet graph of eight hexahedrons

"curl" operator. The nodal equations for EN are equivalent to the nodal FE formulation using scalar potential Ω, whereas loop equations for FN refer to the edge element formulation based on vector potentials $\boldsymbol{A}$.

Table I summarises the equations for both models and shows:

- equations that describe branch fluxes $\boldsymbol{\phi}_b$ in EN and node-to-node magnetic "voltages" $\boldsymbol{u}_{\Omega f}$ for FN; and
- FE equations using the notation of equivalent networks.

In the presented equations, the branch *mmf*s are establish from loop currents (ampere-turns) in the loops around branches. However, when using the FN, it is not necessary to know the branch sources, instead, the loop sources are required. The loop *mmf*s are represented by the currents passing through the loops of FN. For the scalar potential formulation, we define the loop currents to determine the right-hand side

Network	Branch equation	Substitutions	FE equations
Edge	$\boldsymbol{\phi}_b = \boldsymbol{\Lambda}(\boldsymbol{u}_\Omega + \boldsymbol{\Theta}_{be})$	$\boldsymbol{u}_\Omega = \boldsymbol{k}_n\boldsymbol{\Omega}$	$\boldsymbol{k}_n^T\boldsymbol{\Lambda}\boldsymbol{k}_n^T\boldsymbol{\Omega} = -\boldsymbol{k}_n^T\boldsymbol{\Lambda}\boldsymbol{\Theta}_{be}$ [a]
Facet	$\boldsymbol{u}_{\Omega f} = \boldsymbol{R}_\mu\boldsymbol{\phi}_f - \boldsymbol{\Theta}_{bf}$	$\boldsymbol{\phi}_f = \boldsymbol{k}_e\boldsymbol{\phi}_e$	$\boldsymbol{k}_e^T\boldsymbol{R}_\mu\boldsymbol{k}_e\boldsymbol{\phi}_e = \boldsymbol{k}_e^T\boldsymbol{\Theta}_{bf}$ [b]

Notes: $\boldsymbol{\Omega}$ is the vector of nodal potentials, $\boldsymbol{\Lambda}$ is the matrix of branch permeances, $\boldsymbol{\Theta}_{be}$, $\boldsymbol{\Theta}_{bf}$ are the vectors of branch *mmf*s, $\boldsymbol{\phi}_e$ is the vector of loop fluxes; $\boldsymbol{R}_\mu$ is the matrix of branch reluctances, [a] nodal equations of the EN, [b] loop equations of the FN

Table I.
Equations of equivalent magnetic networks

(RHS) vector of the FE equations. For the vector potential approach, the RHS vector can be calculated by using the currents passing through the loops.

For low-frequency problems, we consider two categories of currents: conducting currents and magnetizing currents in regions with permanent magnets. Magnetizing currents are assumed to be known.

III. FE equations of eddy currents

Eddy currents may be described by using the electric scalar potential $\boldsymbol{V}$ or the electric vector potential $\boldsymbol{T}$. The FE equations for the scalar potential formulation and nodal elements represent the nodal equations of the edge electric network. The FE equations for the vector potential $\boldsymbol{T}$ and the edge elements are equivalent to the loop equations of the facet electric network with loops around the element edges. The edges value of $\boldsymbol{T}$ represents the loop currents determining the eddy currents.

The equations for electric network are shown in Table II. The branch equations express the branch currents $\boldsymbol{i}_b$ in EN and the node to node potentials $\boldsymbol{u}_{Vf}$ in FN. In the electric equivalent networks, inter-branch coupling exists.

When formulating equations presented in Table II, the branch *emf*s are expressed by time derivatives of the magnetic fluxes in the loops around the network branches. Therefore, the branch *emf*s in the EN are calculated using the loop fluxes in the facet magnetic network. In the case of the FN analysed using the loop approach, it is not necessary to establish the branch *emf*s $\boldsymbol{e}_{bf}$. The RHS vector $\boldsymbol{e}_{mf}$ in the loop equations is represented by the loop *emf*s, $\boldsymbol{e}_{mf} = \boldsymbol{k}_e^T \boldsymbol{e}_{bf}$. The loop *emf*s in FN are expressed by the time derivatives of fluxes passing through the loops, i.e. the fluxes associated with the branch of the EN.

The disadvantage of the formulation using vector potential $\boldsymbol{T}$ is that the method is only valid for simply connected conductors. However, the electrical machine windings must be considered as multiple connected regions. The FE equations for the classical $\boldsymbol{T}$ formulation refer to loops around the element edges (Figure 3). The loops around the "holes" do not exist. It is therefore necessary to modify the classical $\boldsymbol{T}$ approach and to introduce additional equations describing the loop currents flowing around the "holes" (Demenko *et al.*, 2008). These currents are circuit representation of the edge values of $\boldsymbol{T}_0$ introduced in (Bouissou and Piriou, 1994; Bui *et al.*, 2006).

IV. Equations of winding currents

In the presented approach, the winding currents represent loop currents (Figures 3 and 4). The winding terminals are considered to be out of the region. It is assumed that the terminal voltages are given and the loop *emf*s $\boldsymbol{e}$ produced by external sources are known.

Network	Branch equation	Substitutions	FE equations
Edge	$\boldsymbol{i}_b = \boldsymbol{G}(\boldsymbol{u}_V + \boldsymbol{e}_{be})$	$\boldsymbol{u}_V = \boldsymbol{k}_n \boldsymbol{V}$	$\boldsymbol{k}_n^T \boldsymbol{G} \boldsymbol{k}_n \boldsymbol{V} = -\boldsymbol{k}_n^T \boldsymbol{G} \boldsymbol{e}_{be}$ [a]
Facet	$\boldsymbol{u}_{Vf} = \boldsymbol{R} \boldsymbol{i}_f - \boldsymbol{e}_{bf}$	$\boldsymbol{i}_f = \boldsymbol{k}_e \boldsymbol{i}_e$	$\boldsymbol{k}_e^T \boldsymbol{R} \boldsymbol{k}_e \boldsymbol{i}_e = \boldsymbol{k}_e^T \boldsymbol{e}_{bf}$ [b]

Notes: $\boldsymbol{V}$ is the vector of nodal potentials, $\boldsymbol{G}$ is the matrix of branch conductances, $\boldsymbol{e}_{be}$, $\boldsymbol{e}_{bf}$ are the vectors of branch *emf*s, i.e. is the vectors of loop currents; $\boldsymbol{R}$ is the matrix of branch resistances; [a]nodal equations of the EN; [b]loop equations of the FN

Table II. Equations of equivalent electric networks

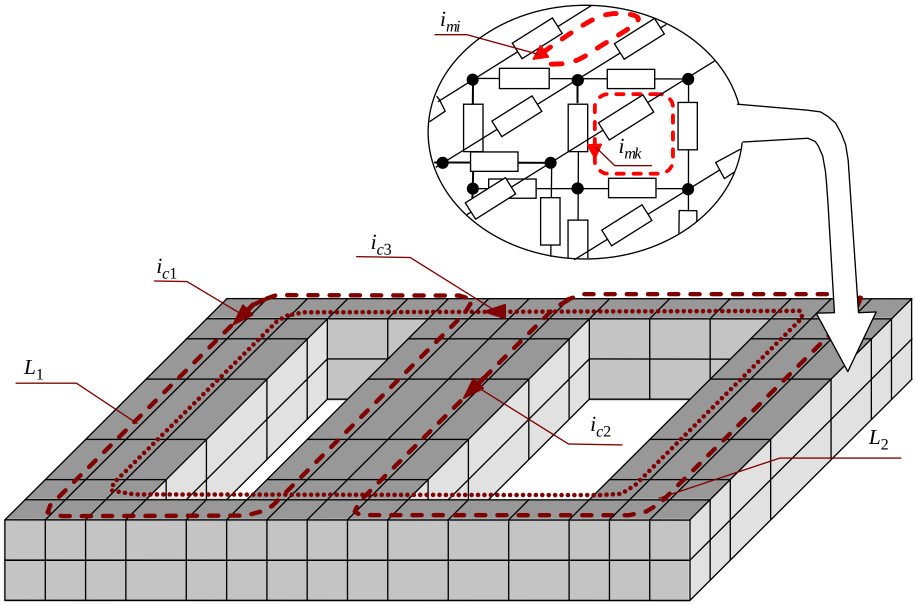


Figure 3.
Multiply connected conducting regions with eddy currents i_m

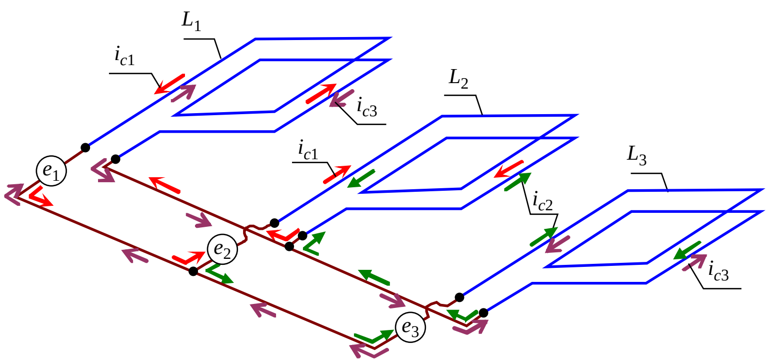
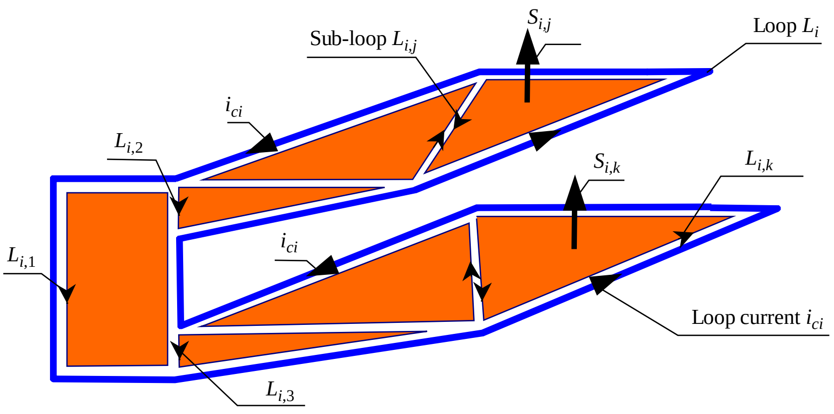


Figure 4.
Loops of three-phase winding composed of stranded conductors

There are two methods for describing winding in the FE space. One method is based on the definition of intersection points between the winding loops and the FE facets (Demenko, 2002). The other more general approach relies on the calculation of intersection points between the FE edges and the surfaces of the winding loops. This method describes the winding in the edge element space (Demenko, 2002).

In Demenko (2002), the winding loops L_i are represented by a set of closed-oriented plane curves $L_{i,j}$ of parametric equations $\boldsymbol{r} = \boldsymbol{r}_{i,j}(t)$ and by planar-oriented surfaces $\boldsymbol{S}_{i,j}$ of parametric equations $\boldsymbol{r} = \boldsymbol{r}_{i,j}(u,v)$ (Figure 5).

The number of intersection points between the edge $\boldsymbol{K}_{p,q}$ (going from P_p to P_q) and surfaces $\boldsymbol{S}_{i,j}$ of L_i represents the entry of $\boldsymbol{N}_e$, i.e. the matrix that describes windings in the edge element space. The matrix entry $N_{eKp,q,i}$ is the difference between the numbers



Source: Demenko (2002)

Figure 5. Sub-loops $L_{i,j}$ of loop L_i with current i_{ci}

of intersection point of positive scalar product $\boldsymbol{S}_{i,j}\,\boldsymbol{K}_{p,q}$ and numbers intersection point of negative scalar product (Demenko, 2002). Therefore, in Figure 6, $N_{eK4,5,I} = 0$.

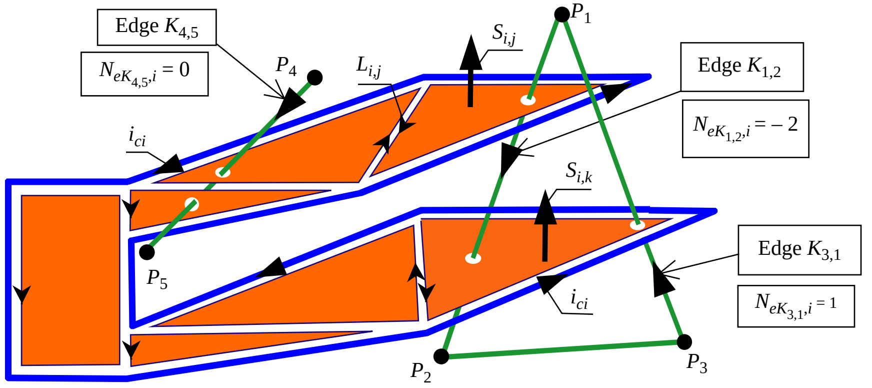
The ampere-turns around the edges represent the branch *mmf*s in magnetic EN. Thus, the vector $\boldsymbol{\Theta}_{be}$ can be defined as follows:

$$\boldsymbol{\Theta}_{be} = \boldsymbol{N}_e \boldsymbol{i}_c. \tag{1}$$

The vector $\boldsymbol{\Theta}_{be}$ can be transformed into the ampere-turns $\boldsymbol{\Theta}_{bf}$ in the loops around the branches of FN (Demenko and Sykulski, 2006). Figure 7 shows this transformation. The transformation matrix $\boldsymbol{K}$ consists of weighted average factors. The product of matrix $\boldsymbol{K}$ and vector $\boldsymbol{\Theta}_{be}$ gives the vector $\boldsymbol{\Theta}_{bf}$ and the branch *mmf*s in FN:

$$\boldsymbol{\Theta}_{bf} == \boldsymbol{K}\boldsymbol{\Theta}_{be} = \boldsymbol{K}\boldsymbol{\Theta}_{be}. \tag{2}$$

In the formulations using potential $\boldsymbol{A}$, the loop sources should be determined (Table I). The loop *mmfs* $\boldsymbol{\Theta}_{mf}$ are obtained by multiplication of the loop matrix $\boldsymbol{k}_e^T$ and the vector $\boldsymbol{\Theta}_{bf}$:



Source: Demenko (2002)

Figure 6. Winding loop in the edge element space

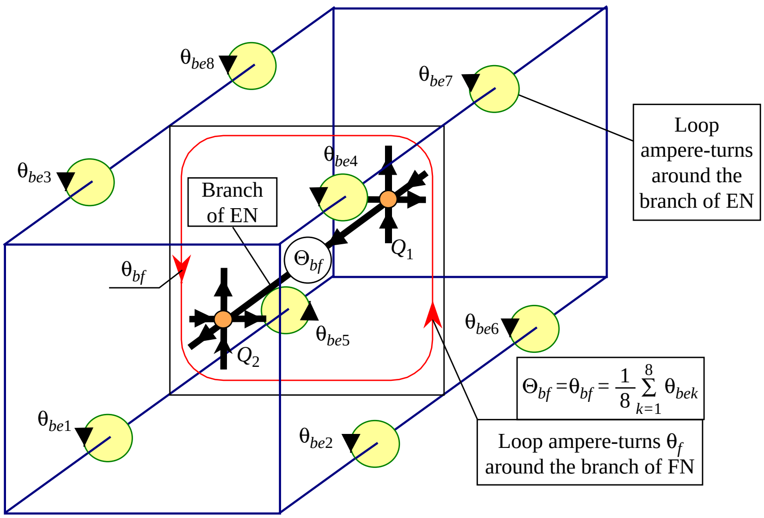


Figure 7.
Transformation of ampere-turns around the branches of the EN into the ampere-turns around the branches of the EN

$$\boldsymbol{\Theta}_{mf} = \boldsymbol{k}_e^T \boldsymbol{\Theta}_{bf} = \boldsymbol{k}_e^T \boldsymbol{K} \boldsymbol{N}_e \boldsymbol{i}_c. \tag{3}$$

The loop *mmfs* $\boldsymbol{\Theta}_{mf}$ may be calculated from the currents (ampere-turns) $\boldsymbol{\Theta}_{me}$ crossing the element faces, i.e. crossing the loops of the EN and representing the facet values of the current density. Matrix $\boldsymbol{k}_e$ transposes the currents in the loops around the edges into the currents in the branch of FN, i.e. into the currents $\boldsymbol{\Theta}_{me}$. Using equation (1), we find that the vector $\boldsymbol{\Theta}_{me}$ is:

$$\boldsymbol{\Theta}_{me} = \boldsymbol{k}_e \boldsymbol{\Theta}_{be} = \boldsymbol{k}_e \boldsymbol{N}_e \boldsymbol{i}_c. \tag{4}$$

The matrix product $\boldsymbol{k}_e \boldsymbol{N}_e$ is equal to the matrix $\boldsymbol{N}_f$ that describes winding loops in the facet element space:

$$\boldsymbol{N}_f = \boldsymbol{k}_e \boldsymbol{N}_e \tag{5}$$

The transposition matrix $\boldsymbol{N}_f$ can be determined by the calculation of intersection points between the loops $L_{i,j}$ and element facets $\boldsymbol{F}_q$. Figure 8 shows the assemblage of matrix $\boldsymbol{N}_f$. The winding loop L_i intersects two times facet $\boldsymbol{F}_q$. The scalar products of $\boldsymbol{F}_q$ and edges of $L_{i,j}$, $L_{i,k}$ are negative. Therefore, the entry $N_{fq,i}$ is equal to -2.

When matrix $\boldsymbol{N}_f$ and currents $\boldsymbol{i}_c$ are given, it is easy to calculate the vector $\boldsymbol{\Theta}_m$. Then, using matrix $\boldsymbol{K}$, the loop *mmf*s $\boldsymbol{\Theta}_{mf}$ in the FN can be found (Demenko and Sykulski, 2006).

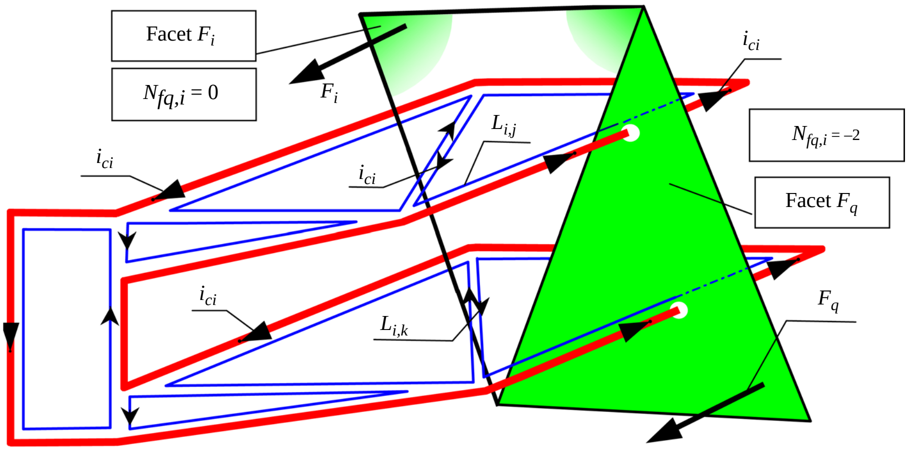
For a given matrix $\boldsymbol{N}_f$, the vector $\boldsymbol{\Theta}_{mf}$ may be expressed as follows:

$$\boldsymbol{\Theta}_{mf} = \boldsymbol{K}^T \ \boldsymbol{\theta}_{me} = \boldsymbol{K}^T \boldsymbol{N}_f \boldsymbol{i}_c. \tag{6}$$

The presented descriptions of a winding can be used for the calculation of flux linkages with the loops L_i. For the EN of the branch fluxes $\boldsymbol{\phi}_b$ the vector $\boldsymbol{\Psi}$ of flux linkages is:

$$\boldsymbol{\Psi} = \boldsymbol{N}_e^T \boldsymbol{\phi}_b. \tag{7a}$$

The description of winding loops in the edge element space is not unique. Matrix $\boldsymbol{N}_e$ is not unique either. The set of surfaces $\boldsymbol{S}_{i,j}$ with the total boundary L_i is not unique. However, the results of the calculation of $\boldsymbol{\Psi}$ are independent of the choice of $\boldsymbol{S}_{i,j}$.



Source: Demenko (2002)

Figure 8.
Winding in the facet element space

In the case of the magnetic vector potential and the facet magnetic network, two formulas can be applied:

$$\boldsymbol{\Psi} = \boldsymbol{N}_e^T \boldsymbol{K}^T \boldsymbol{k}_e \boldsymbol{\phi}_e, \tag{7b}$$

$$\boldsymbol{\Psi} = \boldsymbol{N}_f^T \boldsymbol{K} \boldsymbol{\phi}_e. \tag{7c}$$

A comparison of equation ((7b)-(7c)) with equation (5) yields the identity $\boldsymbol{K}^T \boldsymbol{k}_e = \boldsymbol{K} \boldsymbol{k}_e^T$.

The presented descriptions of the branch and loop *mmfs* and flux linkages are summarized in Table III.

The winding equations can be described in the following unified form:

$$\boldsymbol{R}_m \boldsymbol{i}_c + \mathrm{p}\boldsymbol{\Psi} = \boldsymbol{e}, \tag{8}$$

where $\boldsymbol{R}_m$ is the matrix of loop resistances, $\mathrm{p} = \mathrm{d}/\mathrm{d}t$ and $\boldsymbol{e}$ is the vector of external *emf*s (Figure 4). In the equations for the system in Figure 3, the vector $\boldsymbol{e}$ is equal to the voltages produces by the eddy currents;, i.e. $\boldsymbol{e} = -\boldsymbol{N}_e^T \boldsymbol{k}_e^T \boldsymbol{R} \boldsymbol{k}_e \boldsymbol{i}_e$.

V. Equations of coupled magneto-electric model

In order to form the complete field model of the machine, the link between the magnetic field and the eddy currents must be considered. It has been shown (Demenko and Sykulski, 2006) that branch sources in the FN are established from loop quantities in

Table III.
Descriptions of mmfs and flux linkages electric networks

Network	Branch *mmf*s	Loop *mmf*s	Flux linkages
Edge	$\boldsymbol{\Theta}_{be} = \boldsymbol{N}_e \boldsymbol{i}_c$	$\boldsymbol{\Theta}_{me} = \boldsymbol{N}_f \boldsymbol{i}_c$	$\boldsymbol{\Psi} = \boldsymbol{N}_e^T \boldsymbol{\phi}_b$
Facet	$\boldsymbol{\Theta}_{bf} = \boldsymbol{K}\boldsymbol{N}_e \boldsymbol{i}_c$	$\boldsymbol{\Theta}_{mf} = \boldsymbol{K}^T \boldsymbol{N}_f \boldsymbol{i}_c$	$\boldsymbol{\Psi} = \boldsymbol{N}_f^T \boldsymbol{K} \boldsymbol{\phi}_e$

Notes: The matrix $\boldsymbol{N}_e$ describes windings in the edge element space, the matrix $\boldsymbol{N}_f$ describes windings in the edge element space $\boldsymbol{i}_c$ is the vector of currents in winding loops $\boldsymbol{K}^T \boldsymbol{k}_e = \boldsymbol{K} \boldsymbol{k}_e^T$, $\boldsymbol{N}_f = \boldsymbol{k}_e \boldsymbol{N}_e$

the EN, and – by symmetry – branch sources in the EN are found from loop quantities in the FN. The branch *mmf*s $\boldsymbol{\Theta}_{be}$ in the EN correspond to loop currents. Branch *emf*s $\boldsymbol{e}_{be}$ in the EN are found as time derivatives of loop fluxes $\boldsymbol{\phi}_e$ in the FN. Using the symbols in Tables I and II, the branch sources of EN can be written as:

$$\Theta_{be} = \boldsymbol{i}_e, \quad \boldsymbol{e}_{be} = -\frac{\mathrm{d}\boldsymbol{\phi}_e}{\mathrm{d}\boldsymbol{t}}. \tag{9}$$

The loop *mmf* is equivalent to the current passing through the loop of the magnetic network, thus the loop *mmf*s $\boldsymbol{\Theta}_{mf}$ in the FN correspond to the branch currents $\boldsymbol{i}_b$ in the EN (Table II). In the FN models of the eddy current regions, the loop *emf*s may be found by taking time derivatives of the branch fluxes in the magnetic network passing through the loops of the electric network. The loop sources in FN can be expressed by:

$$\boldsymbol{\Theta}_{mf} = \boldsymbol{i}_b = \boldsymbol{G}(\boldsymbol{k}_n \boldsymbol{V} - \mathrm{p}\boldsymbol{\phi}_e), \tag{10a}$$

$$\boldsymbol{e}_{mf} = -\mathrm{p}\boldsymbol{\phi}_b = -\mathrm{p}(\boldsymbol{\Lambda}(\mathrm{k}_n\boldsymbol{\Omega} + \boldsymbol{\Theta}_{be})). \tag{10b}$$

The field sources in the FN can be also calculated using the relations presented in section IV. The winding loops should be considered as eddy current loops:

$$\boldsymbol{\Theta}_{mf} = \boldsymbol{k}_e^T \boldsymbol{K} \boldsymbol{i}_e, \quad \boldsymbol{e}_{mf} = -\mathrm{p}\boldsymbol{K}^T \boldsymbol{k}_e \boldsymbol{\phi}_e. \tag{11}$$

Based on the above-presented equations, the field model of the electrical machine is constructed.

The FE equations for $\Omega - \boldsymbol{T} - \boldsymbol{T}_0$ formulation are represented by nodal equations of the edge magnetic network coupled with the loop equations that describe eddy currents in the electric FN and currents in winding loops. These equations can be written in the following matrix form:

$$\begin{bmatrix} \boldsymbol{k}_n^T \boldsymbol{\Lambda} \boldsymbol{k}_n & \boldsymbol{k}_n^T \boldsymbol{\Lambda} & \boldsymbol{k}_n^T \boldsymbol{\Lambda} \boldsymbol{N}_e \\ \mathrm{p}\boldsymbol{\Lambda} \boldsymbol{k}_n & \boldsymbol{R}_e + \mathrm{p}\boldsymbol{\Lambda} & (\boldsymbol{R}_e + \mathrm{p}\boldsymbol{\Lambda})\boldsymbol{N}_e \\ \mathrm{p}\boldsymbol{N}_e^T \boldsymbol{\Lambda} \boldsymbol{k}_n & \boldsymbol{N}_e^T(\mathrm{p}\boldsymbol{\Lambda} + \boldsymbol{R}_e) & \boldsymbol{R}_m + \boldsymbol{N}_e^T \mathrm{p}\boldsymbol{\Lambda} \boldsymbol{N}_e \end{bmatrix} \begin{bmatrix} \boldsymbol{\Omega} \\ \boldsymbol{i}_e \\ \boldsymbol{i}_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{k}_n^T \boldsymbol{\Lambda} \boldsymbol{\theta}_b \\ 0 \\ \boldsymbol{e} \end{bmatrix}. \tag{12}$$

Here, $\boldsymbol{R}_e$ is the matrix of loop resistances for loops with eddy currents, $\boldsymbol{R}_e = \boldsymbol{k}_e^T \boldsymbol{R} \boldsymbol{k}_e$, and $\boldsymbol{\theta}_b$ is the vector of additional branch *mmf*s in the permanent magnet region. These *mmf*s represent the edge values of magnetization vector. In the above equations, vector $\boldsymbol{i}_c$ describes the winding currents and the currents in the loops around the "holes" in the region with eddy currents.

It seems that for the eddy-current calculation the most convenient is the $\boldsymbol{A} - V - \boldsymbol{T}_0$ formulation. This formulation is equivalent to the loop analysis of the facet magnetic network, coupled with nodal analysis of the EN for eddy currents and with the loop description of winding with stranded conductors. The FE equations for the $\boldsymbol{A} - V - \boldsymbol{T}_0$ formulation are:

$$\begin{bmatrix} \boldsymbol{k}_e^T \boldsymbol{R}_\mu \boldsymbol{k}_e + \boldsymbol{G}\mathrm{p} & -\boldsymbol{G}\boldsymbol{k}_n & -\boldsymbol{K}^T \boldsymbol{N}_f \\ -\mathrm{p}\boldsymbol{k}_n^T \boldsymbol{G} & \boldsymbol{k}_n^T \boldsymbol{G}\boldsymbol{k}_n & 0 \\ \mathrm{p}\boldsymbol{N}_f^T \boldsymbol{K} & 0 & \boldsymbol{R}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi}_e \\ \boldsymbol{V} \\ \boldsymbol{i}_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{\theta}_m \\ 0 \\ \boldsymbol{e} \end{bmatrix}, \tag{13}$$

where $\boldsymbol{\theta}_m$ is the vector of loop *mmf*s in the regions with permanent magnets; $\boldsymbol{\theta}_m = \boldsymbol{k}_e^T \boldsymbol{K} \boldsymbol{\theta}_b$.

If the $\boldsymbol{A}$–$\boldsymbol{T}$–$\boldsymbol{T}_0$ formulation is applied, the FE equations represent the loop equations for the magnetic and electric FNs. These equations can be expressed by:

$$\begin{bmatrix} \boldsymbol{k}_e^T \boldsymbol{R}_\mu \boldsymbol{k}_e & -\boldsymbol{k}_e^T \boldsymbol{K} & -\boldsymbol{K}^T \boldsymbol{N}_f \\ \mathrm{p}\boldsymbol{K}^T \boldsymbol{k}_e & \boldsymbol{R}_e & \boldsymbol{k}_e^T \boldsymbol{R}\boldsymbol{N}_f \\ \mathrm{p}\boldsymbol{N}_f^T \boldsymbol{K} & \boldsymbol{N}_f^T \boldsymbol{R}\boldsymbol{k}_e & \boldsymbol{R}_m \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi}_e \\ \boldsymbol{i}_e \\ \boldsymbol{i}_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{\theta}_m \\ 0 \\ \boldsymbol{e} \end{bmatrix}. \tag{14}$$

VI. Rotor motion simulation

The FE methods considering rotor motion can be divided into two categories:

(1) techniques with the fixed grid independent of the moving region position; and

(2) the techniques with the moving grid (Sadowski *et al.*, 1992; Trowbridge and Sykulski, 2006; Williamson, 1994).

The fixed grid methods have been successfully applied in the analysis of the systems with homogenous moving part and constant speed.

The moving grid methods are more general. In these techniques, the grid is divided into two parts: the moving part associated with the rotor and fixed part associated with the stator. Between these parts, an interconnecting band or slip surface is created. The most popular methods for coupling the fixed and moving part through the band or the slip surface can be easily understood by using the notation of equivalent networks.

In the scalar potential method, the changes of the rotor position are modelled by the changes of the nodal permeance matrix $\boldsymbol{k}_n^T \boldsymbol{\Lambda} \boldsymbol{k}_n^T$. For the vector potential method corresponding to the loop analysis of the FN, the changes of rotor position α are represented by the changes of the loop reluctance matrix $\boldsymbol{k}_e^T \boldsymbol{R}_\mu \boldsymbol{k}_e$. At first, we will study the moving grid method for the formulation using the magnetic vector potential $\boldsymbol{A}$. In this formulation, the changes can be related to the factors of the product $\boldsymbol{k}_e^T \boldsymbol{R}_\mu \boldsymbol{k}_e$:

- the structural matrix $\boldsymbol{k}_e$;
- the matrix $\boldsymbol{R}_\mu$ of branch reluctances; or
- both the matrix $\boldsymbol{k}_e$ and $\boldsymbol{R}_\mu$.

In the method (1), we represent the discrete position of rotor by the distribution of nonzero elements of the loop matrix $\boldsymbol{k}_e$. As a result, we obtain a set of matrices $\boldsymbol{k}_e$ for successive rotor positions. This set represents the data points for interpolation functions expressing the dependence of $\boldsymbol{k}_e$ on the angle α which describes the rotor position (Demenko, 1996). It can be proven that the use trigonometric interpolation is equivalent to the application of harmonic weighting functions (De Gersem *et al.*, 2006). When a suitable interpolation of matrix $\boldsymbol{k}_e$ is applied the method of category (1) guarantees high accuracy. However, due to the increase of the density of the matrix $\{\boldsymbol{k}_e(\alpha)\}^T \boldsymbol{R}_\mu \boldsymbol{k}_e(\alpha)$, the procedure of solving the FE equations becomes complicated and time-consuming. Also, in the case of methods of category (2) with $\boldsymbol{R}_\mu = \boldsymbol{R}_\mu(\alpha)$, the matrix of loop reluctances is dense which results in an increase of computation time. The most representative method of category (2) is the air-gap element method.

Owing to its simplicity, the methods of category (3), i.e. the moving band approach with remeshing of the FE network, belong to the most popular methods (Davat *et al.*, 1985; Tsukerman, 1992).

The presented method for movement simulation in combination with vector potential formulation can be easy adopted for scalar potential method. Changes of matrices $\boldsymbol{R}_\mu$, $\boldsymbol{k}_e$ are represented by changes of the matrices $\boldsymbol{\Lambda}$, $\boldsymbol{k}_n$ in the scalar potential methods.

VII. Electromagnetic torque

The electromagnetic torque is determined by the virtual work principle. The formulas can be divided into two categories:

(1) the force density formulas, e.g. Lorenz formula, the method of magnetizing currents; and

(2) stress tensor formulas, e.g. the Maxwell stress tensor formula (Ren, 1994; Sadowski *et al.*, 1992).

Of course, for the exact solution of Maxwell's equations, the formulas of both categories give identical results. The position of integration surface has an effect on the result of the Maxwell stress tensor. Very often, in the FE models, the electromagnetic torque is calculated by the virtual work principle (Coulomb and Meunier, 1984; Demenko, 1996; Demenko and Stachowiak, 2008).

In accordance with the virtual work principle, for scalar potential method, the torque is equal to the magnetic coenergy derivative versus the virtual displacement. An interpolation function can be applied to describe this derivative:

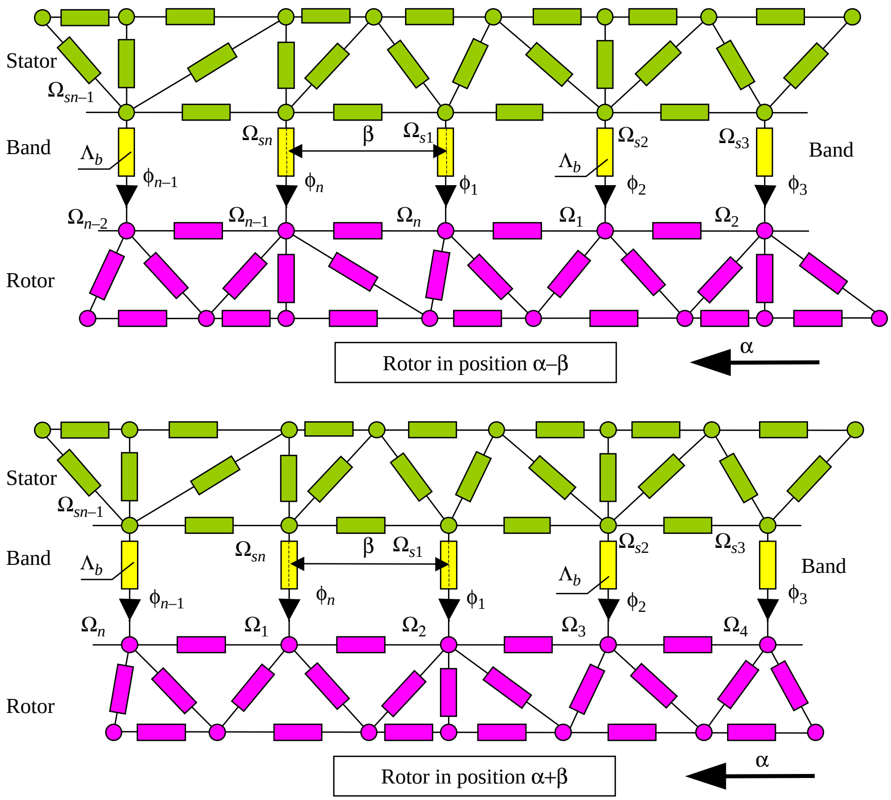
$$T(\alpha) = \left.\frac{\partial W_c(\alpha + \Delta\alpha)}{\partial(\Delta\alpha)}\right|_{\Delta\alpha=0} = \frac{W_c(\alpha + \beta) - W_c(\alpha + \beta)}{2\beta}, \tag{15}$$

where $W_c(\alpha \pm \beta)$ is the magnetic coenergy for the rotor position $\alpha \pm \beta$ (Figure 9). From equation (17), using symbols in Table I, we obtain:

$$T(\alpha) = \frac{1}{4\beta}\boldsymbol{\Omega}^T\left[\left.\left(\boldsymbol{k}_n^T\boldsymbol{\Lambda}\boldsymbol{k}_n\right)\right|_{\alpha+\beta} - \left.\left(\boldsymbol{k}_n^T\boldsymbol{\Lambda}\boldsymbol{k}_n\right)\right|_{\alpha-\beta}\right]\boldsymbol{\Omega}. \tag{16}$$

For the vector potential formulation, the magnetic energy derivative versus the virtual displacement is considered (Demenko, 1998) and:

$$T(\alpha) = -\frac{1}{4\beta}\boldsymbol{\phi}_e^T\left[\left.\left(\boldsymbol{k}_e^T\boldsymbol{R}_\mu\boldsymbol{k}_e\right)\right|_{\alpha+\beta} - \left.\left(\boldsymbol{k}_e^T\boldsymbol{R}_\mu\boldsymbol{k}_e\right)\right|_{\alpha-\beta}\right]\boldsymbol{\phi}_e. \tag{17}$$



Source: Demenko and Stachowiak (2008)

Figure 9.
A part of magnetic EN for two discrete rotor positions

Particularly noteworthy are the methods adapted to the applied techniques of motion simulation (Coulomb and Meunier, 1984; Demenko, 1996). For example, if in the procedure of motion simulation, $\boldsymbol{R}_\mu = \boldsymbol{R}_\mu(\alpha)$ then:

$$T(\alpha) = -0.5\boldsymbol{\phi}_e^T \left\{ \boldsymbol{k}_e^T \left(\frac{\mathrm{d}\boldsymbol{R}_\mu}{\mathrm{d}\alpha} \right) \boldsymbol{k}_e \right\} \boldsymbol{\phi}_e. \quad (18)$$

This formula and equations (16) and (17) can be considered as the FE representation of Maxwell stress formula with integration surface related to the band between the stator and rotor.

VIII. Field-circuit model

In this paper, the term field-circuit model of electrical machine is related to the approaches expressing the flux linkages $\boldsymbol{\Psi}$ with the windings by two components:

(1) components defined by field quantities; and

(2) component represented by inductances (Demenko and Hameyer, 2008; Lange *et al.*, 2008).

The expressions in Table III become:

$$\boldsymbol{\Psi} = \boldsymbol{N}_f^T \boldsymbol{K} \boldsymbol{\phi}_e + \boldsymbol{L}_e \boldsymbol{i}_c \tag{19a}$$

$$\boldsymbol{\Psi} = \boldsymbol{N}_e^T \boldsymbol{\phi}_b + \boldsymbol{L}_e \boldsymbol{i}_c, \tag{19b}$$

where $\boldsymbol{L}_e$ is the matrix of equivalent inductances.

The field-circuit coupled model is applied when the magnetic field is assumed to be 2D. In this model, the matrix $\boldsymbol{L}_e$ describes the inductances of the end-winding.

To obtain the equations of the field-circuit coupled model, the matrix $\boldsymbol{R}_m$ in equations (12)-(14) should be replaced by the sum $\boldsymbol{R}_m + \mathrm{p}\ \boldsymbol{L}_e$.

Usually, in the modelling of electrical machines equivalent circuits are applied. The equivalent circuits are formed by the application of current transformations;, e.g. Clarke and Park transformation (Jones, 1967). The inductances of equivalent transformed system can be directly calculated using the field model. For elementary values of currents $\boldsymbol{i}_T$, the currents $\boldsymbol{i}_c$ in winding loops are calculated, $\boldsymbol{i}_c = \boldsymbol{k}^T \boldsymbol{i}_T$, where $\boldsymbol{k}$ is the transformation matrix. Then, for the currents $\boldsymbol{i}_c$ the FE equations are solved and the vectors $\boldsymbol{\Psi}_T$ of flux linkages for transformed system are determined, $\boldsymbol{\Psi}_T = \boldsymbol{k\Psi}$. These vectors represent the inductances of the transformed circuit model.

IX. Conclusions

The paper presents the field and the field-circuit models of electrical machines. The studied approaches have been elaborated to simulate the machine's behaviour with the presented models (De Gersem *et al.*, 1998, 2000; Lahaye *et al.*, 2002). The developments of these approaches and in the methods to solve large systems of equations enable the application of the presented models to many practical applications of technical significances.

Field simulations are applied for the machine's diagnostic (Weili *et al.*, 2007). However, in diagnostics the field models are not as popular as for machine design. Recently, field models become more and more popular in the analysis and synthesis of electrical machine control, even though the control methods are based on the classical circuit approaches. Mostly, the field methods are used to calculate the parameters of the equivalent circuit of considered control system (Brulé and Tounzi, 2000: Di Napoli and Santini, 2000). The field methods can be especially helpful in the case of sensorless control when an accurate description of the machine parameters is required.

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