SUPG 3D vector potential formulation for electromagnetic braking simulations

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***Abstract*— The calculation of motion-induced eddy currents in massive conductors yields a 3D convection-diffusion problem. Up-winding and SUPG formulations are well-established methods to obtain stable discretizations of the scalar convection-diffusion equations in the case of singular perturbation, but there is very little reported experience with the stability of convection in the vector case, i.e. in electromagnetism. Numerical experiments with the up-winding method proposed by Xu et al. [1] has proven its efficiency to be limited, and an alternative approach based on a consistent discretization within the finite element Galerkin context of the material derivative implied by the convection phenomenon is proposed.**

I. Introduction

The problem of electromagnetic braking can solved by a quasi-stationary approach by discretizing the model in the rest frame. According to e.g. [1], the governing equations are $\mathbf{b} = \nabla \times \mathbf{a}$, $\mathbf{e} = -\partial_t \mathbf{a} - \nabla V$ and $\nabla \times \mathbf{h} = \mathbf{j}$ with the material constitutive relations $\mathbf{b} = \mu \mathbf{h}$ and

$$\mathbf{j} = \sigma(\mathbf{e} + \mathbf{v} \times \mathbf{b}), \tag{1}$$

where the velocity field $\mathbf{v}$ is different from zero in the moving domain. There are several interpretations found in literature for formulae like (1). The interpretations of Xu [1], Bossavit [5], Thorne [6] and Van Bladel [7] will be discussed and compared in the full paper.

On the other hand, it would seem natural, by analogy with mechanics to work with Eulerian coordinates, i.e. to replace the equation

$$\sigma \partial_t \mathbf{a} + \mathrm{curl}\, \nu \mathrm{curl}\, \mathbf{a} = \mathbf{j}_s \tag{2}$$

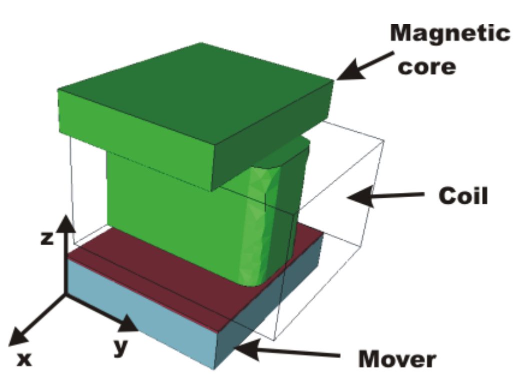

Fig. 1. Geometry of the 3D braking system.

valid when there is no motion, with the equation

$$\sigma D_t \mathbf{a} + \mathrm{curl}\, \nu \mathrm{curl}\, \mathbf{a} = \mathbf{j}_s \tag{3}$$

where D_t denotes the convective derivative. This would be a justification for importing stabilization SUPG scheme from Mechanics, in particular computational fluid dynamics [2], [3], into electromagnetic problems. We therefore have first to answer the question : Is electromagnetic braking a convection problem ?

Given a placement map $p_t : M \mapsto N$, one can define the co-moving time derivative as the derivative operator that fulfills

$$\partial_t \int_\Omega \alpha = \int_\Omega \mathcal{L}_\mathbf{v}\, \alpha.$$

The co-moving time derivative of differential forms of various degrees write as follows

$$\begin{aligned} \mathcal{L}_\mathbf{v}\, f &= \partial_t f + \mathbf{v} \cdot (\mathrm{grad}\, f) && (4)\\ \mathcal{L}_\mathbf{v}\, \mathbf{a} &= \partial_t \mathbf{a} + \mathrm{grad}\, (\mathbf{a} \cdot \mathbf{v}) - \mathbf{v} \times \mathrm{curl}\, \mathbf{a} && (5)\\ \mathcal{L}_\mathbf{v}\, \mathbf{b} &= \partial_t \mathbf{b} + \mathrm{curl}\, (\mathbf{b} \times \mathbf{v}) + \mathbf{v}\ \mathrm{div}\, \mathbf{b} && (6)\\ \mathcal{L}_\mathbf{v}\, \rho &= \partial_t \rho + \mathrm{div}\, (\rho \mathbf{v}) && (7) \end{aligned}$$

in terms of classical vector and tensor analysis operators. The co-moving derivative of 0-forms (5) and 3-forms (7) are commonly used in computational fluid dynamics where they are called (amongst many other names) convective derivative, $\mathcal{L}_\mathbf{v} \equiv D_t$.

The electric field writes in the absence of motion $\mathbf{e} = -\partial_t \mathbf{a} - \mathrm{grad}\, u$ becomes in an Eulerian representation,

$$\begin{aligned} \mathbf{e} &= -\mathcal{L}_\mathbf{v}\, \mathbf{a} - \mathrm{grad}\, u \\ &= -(\partial_t \mathbf{a} + \mathrm{grad}\, (\mathbf{a} \cdot \mathbf{v}) - \mathbf{v} \times \mathrm{curl}\, \mathbf{a}) - \mathrm{grad}\, u \end{aligned}$$

in the presence of motion. One observes the introduction through the co-moving time derivative of the classical $\mathbf{v} \times \mathbf{b}$ (1). But one observes also a motion induced correction to the electric scalar potential, $\mathrm{grad}\, (\mathbf{a} \cdot \mathbf{v})$, which is not considered in the classical definition of motion induced eddy currents. The electric field can actually be rewritten

$$\mathbf{e}' = -\partial_t \mathbf{a}' - \mathrm{grad}\, u'$$

with the auxiliary fields : $\mathbf{e}' = \mathbf{e} - \mathbf{v} \times \mathrm{curl}\, \mathbf{a}$, $\mathbf{a}' = \mathbf{a}$ and $u' = u + \mathbf{a} \cdot \mathbf{v}$. The co-moving time derivative appears thus to be related with the Lorentz invariance of Maxwell's equations, as can be shown in a slightly extended theoretical context.

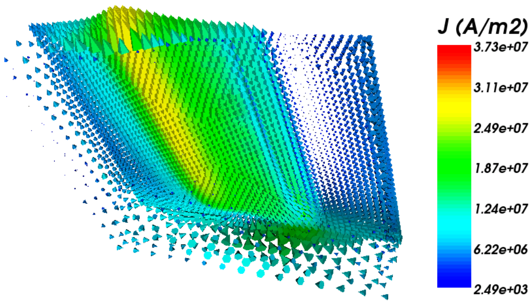

Fig. 2. Current density computed in the mover.

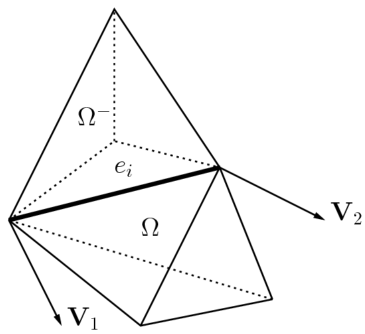

Fig. 3. Arbitrary finite element Ω and its upwind element Ω^-.

II. Xu's SUPG approach

Assuming a stationary process, $\partial_t \mathbf{a} = 0$, the scheme proposed by Xu et al in [1], directly inspired from computational fluid dynamics, is as follows :

$$\int_\Omega \mu^{-1} \mathrm{curl}\, \mathbf{a} \cdot \mathrm{curl}\, \mathbf{w}\, \mathrm{d}\Omega + \int_{\partial\Omega} (\mathbf{w} \times \mu^{-1} \mathrm{curl}\, \mathbf{a}) \cdot \mathbf{n}\, \mathrm{d}\partial\Omega = - \int_\Omega \sigma\, (-\mathbf{v} \times \mathrm{curl}\, \mathbf{a} + \mathrm{grad}\, u) \cdot \mathbf{w}\, \mathrm{d}\Omega$$

In analogy with the upwind scalar shape functions (with a free parameter τ) :

$$w = w_0 + \tau \frac{\mathbf{v} \cdot \mathrm{grad}\, w_0}{||\mathbf{v}||^2} \quad , \quad \tau = \frac{\mathbf{v} h}{2} (\coth \frac{Pe}{2} - \frac{2}{Pe})$$

upwind vector shape functions are defined as

$$\mathbf{w} = \mathbf{w}_1 - \tau\sigma \mathbf{v} \times \mathrm{curl}\, \mathbf{w}_1$$

Numerical 3D simulations done with this approach, Fig. 2, show that a stabilization effect is indeed observed, but leaves still a severe limitation on the convection speed $\mathbf{v}$.

III. A geometrical scheme

In their report [4], Heumann and Hiptmair have successfully exploited differential geometry concepts to obtain a geometrical discretization of the convection operator in 2D. Their approach is based on the extrusion operator introduced by Bossavit [5]. The purpose of this paper is to generalize their 2D scheme to 3D finite element computations.

Consider the situation depicted in Fig. 3. Let Ω be an arbitrary finite element in a 3D mesh. A particular edge of that element e_i is considered, at both ends of which the velocity vector has been represented. This edge e_i represented in the figure is such that its upwind extrusion lays outside Ω, i.e. in a neighbour element $\Omega^- \neq \Omega$. For the FE discretisation, one has to evaluate :

$$\int_\Omega \sigma\, \mathcal{L}_\mathbf{v}\, \mathbf{a} \cdot \omega_l^e\, \mathrm{d}\Omega = \sum_i \left\{ \int_{e_i} \mathcal{L}_\mathbf{v}\, \mathbf{a}(\Omega_i^-) \right\} \int_\Omega \sigma \omega_i^e \cdot \omega_l^e\, \mathrm{d}\Omega, \tag{8}$$

where ω_l^e is and edge-based trial function. But the tangential component of $\mathcal{L}_\mathbf{v}\, \mathbf{a}$ is not continuous. The fact that the derivative $\mathcal{L}_\mathbf{v}$ is a limiting process involving the upwind extrusion of e_i, which lays in the upwind element Ω^- relative to the edge e_i under consideration, imposes thus to evaluate the circulation of $\mathcal{L}_\mathbf{v}\, \mathbf{a}$ in that element, i.e. $\int_{e_i} \mathcal{L}_\mathbf{v}\, \mathbf{a}(\Omega_i^-)$. In the evaluation of the residual (8) of the finite element Ω, the adjacent upwind elements Ω_i^- plays thus a role. This is incompatible with the classical element by element assembly of FE elementary matrices, hence an implementation difficulty.

An algorithm to evaluate $\int_{e_i} \mathcal{L}_\mathbf{v}\, \mathbf{a}(\Omega_i^-)$ will be described in the full paper. This expression has 3 terms (5). In particular, it will be shown that

$$\int_{e_i} \mathbf{v} \times \mathrm{curl}\, \mathbf{a} = \sum_{jl} A_j \frac{V_1^l + V_2^l}{2} T_{ilj}$$

with

$$T_{ilj} = \mathrm{i}_{e_i} \mathrm{i}_{e_l} \mathrm{curl}\, \omega_j^e(\mathbf{x})$$

a constant matrix with ± 1 and 0 elements that only depends on the topology of the tetrahedron.

IV. Conclusion

We have discussed the validity of introducing convection based concepts in context of electromagnetism, and shown that this is done by the co-moving time derivative. The concept of extrusion yields a geometrical upwind scheme without free parameter that can however not be assembled element by element. More numerical results will be given in the full paper.

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