SUPG 3D vector potential formulation for electromagnetic braking simulations

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***Résumé* — The calculation of motion-induced eddy currents in massive conductors yields a 3D convection-diffusion problem. Up-winding and SUPG formulations are well established methods to obtain stable discretizations of the scalar convection-diffusion equations in the case of singular perturbation, but there is very little reported experience with the stability of convection in the vector case. Numerical experiments with the up-winding method proposed by Xu et al. [1] are presented and analysed. Their scheme is interpreted in geometrical terms, and an alternative approach based on a consistent discretization within the finite element Galerkin context of the Lie derivative implied by the convection phenomenon is proposed.**

I. INTRODUCTION

Eddy currents induced in moving massive conductors have important engineering applications, such as electromagnetic braking and non-destructive testing. As the velocity of moving conductors increases, solutions by standard Galerkin Finite Element (FE) formulations usually develop spurious oscillations. The Streamline Upwind Petrov Galerkin (SUPG) formulation is a well-established stabilization method in computational fluid dynamics [2], whose principle is to mimic the flow of information by giving more importance to data that comes from the upwind region, whereas data that comes from the downwind region is given lesser importance or even neglected [3]. However, there are very few publications, besides the paper by Xu et al. [1], describing the application of the SUPG principle to 3D electromagnetic problems. After implementation, it turns out that this scheme is very sensitive to the discretization. A deeper theoretical analysis is proposed in this paper, based on the work by Heumann [4], who has shown, in the scalar case, that in a variational setting the discrete Lie derivative is identical to a Galerkin method using some inexact first order upwind quadrature.

II. VECTOR POTENTIAL 3D FORMULATIONS

Let $W^p(\Omega), p = 0, 1, 2, 3$ be the sets of differential forms of degree p defined on Ω. The equations to solve are

$$\begin{aligned} \operatorname{curl} \mathbf{h} &= \mathbf{j} && (1)\\ \mathbf{e} &= -\mathcal{L}_{\mathbf{v}}\, \mathbf{a} - \operatorname{grad} u && (2)\\ \mathbf{j} &= \sigma \mathbf{e} && (3)\\ \mathbf{b} &= \operatorname{curl} \mathbf{a} = \mu \mathbf{h} && (4) \end{aligned}$$

with $\mathbf{h} \in W^1(\Omega)$ the magnetic field, $\mathbf{j} \in W^2(\Omega)$ the current density, $\mathbf{a} \in W^1(\Omega)$ the magnetic vector potential and $u \in W^0(\Omega)$ the electric scalar potential. In (2), $\mathcal{L}_{\mathbf{v}}$ represents the co-moving time derivative, also called material derivative, associated with the flow velocity $\mathbf{v}$. Taking into account the fact that $\mathbf{a}$ is 1-form, the co-moving time derivative can be expressed in terms of classical vector analysis operators as

$$\mathcal{L}_{\mathbf{v}}\, \mathbf{a} = \partial_t \mathbf{a} + \operatorname{grad} (\mathbf{v} \cdot \mathbf{a}) - \mathbf{v} \times \operatorname{curl} \mathbf{a}. \tag{5}$$

Substituting (2)-(4) into (1) yields

$$\operatorname{curl} \mu^{-1} \operatorname{curl} \mathbf{a} = -\sigma \left(\mathcal{L}_{\mathbf{v}}\, \mathbf{a} + \operatorname{grad} u \right). \tag{6}$$

The Petrov-Galerkin weak formulation consists in multiplying the latter with a trial function $\mathbf{w}$ and stating, after integration by parts, that the equation

$$\int_\Omega \mu^{-1} \operatorname{curl} \mathbf{a} \cdot \operatorname{curl} \mathbf{w}\, d\Omega + \int_{\partial\Omega} (\mathbf{w} \times \mu^{-1} \operatorname{curl} \mathbf{a}) \cdot \mathbf{n}\, d\partial\Omega = - \int_\Omega \sigma \left(\mathcal{L}_{\mathbf{v}}\, \mathbf{a} + \operatorname{grad} u \right) \cdot \mathbf{w}\, d\Omega \tag{7}$$

must hold for all $\mathbf{w}$ chosen in a suitable function space. Substituting (5), one obtains finally

$$\int_\Omega \mu^{-1} \operatorname{curl} \mathbf{a} \cdot \operatorname{curl} \mathbf{w}\, d\Omega + \int_{\partial\Omega} (\mathbf{w} \times \mu^{-1} \operatorname{curl} \mathbf{a}) \cdot \mathbf{n}\, d\partial\Omega = - \int_\Omega \sigma \left(\partial_t \mathbf{a} - \mathbf{v} \times \operatorname{curl} \mathbf{a} + \operatorname{grad} (u + \mathbf{a} \cdot \mathbf{v}) \right) \cdot \mathbf{w}\, d\Omega. \tag{8}$$

III. SUPG SCHEME

Let $W_h^p(\Omega)$ be the discrete equivalents for the $W^p(\Omega)$ sets defined above, i.e. the Whitney form sets. The Streamline Upwind Petrov Galerkin formulation for Navier-Stokes equation consists in choosing the trial function as

$$w = w_0 + \tau \frac{\mathbf{v} \cdot \operatorname{grad} w_0}{||\mathbf{v}||^2} \quad , \quad \tau = \frac{\mathbf{v} h}{2} (\coth \frac{Pe}{2} - \frac{2}{Pe}) \tag{9}$$

with $w_0 \in W_h^0(\Omega)$ and Pe the local Peclet number of the mesh.

In order to obtain a stabilized formulation for the electromagnetic problem under consideration, an upwinding method for differential forms of degree 1 must be determined. There is however very little publications on this topic. In [1], a vector generalization of (9) is proposed

$$\mathbf{w} = \mathbf{w}_1 - \tau \sigma \mathbf{v} \times \operatorname{curl} \mathbf{w}_1 \tag{10}$$

with $\mathbf{w}_1 \in W_h^1(\Omega)$ and the local Peclet number and the mesh size in the direction of the flow defined as

$$Pe = \mu \sigma \mathbf{v} h_v \quad , \quad 2 h_v = \left(\sum_n \frac{\mathbf{v}}{v} \cdot \operatorname{grad} \mathbf{w_0} \right)^{-1} . \tag{11}$$

This proposition comes however with no theoretical justification.

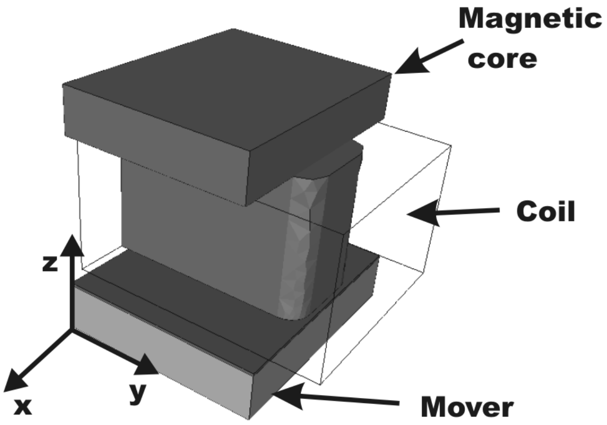

Fig. 1. Geometry of the finite element model for the electromagnetic brake. Due to the symmetry, only one half pole is modelled.

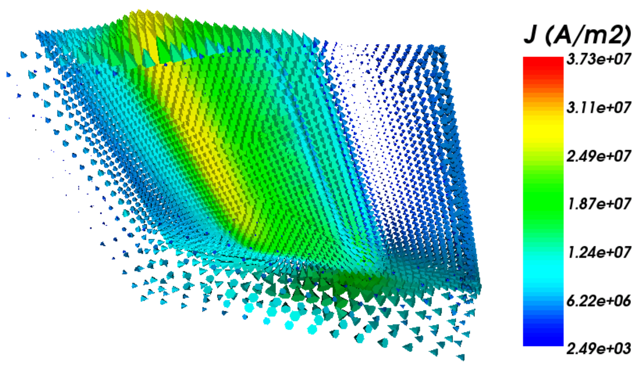

Fig. 2. Eddy currents in the mover seen from below.

IV. NUMERICAL RESULTS

The SUPG scheme proposed by Xu et al has been applied to the computation of the electromagnetic braking effect exerted on an infinitely long steel mover by a standard linear motor stator. The mover is a massive piece of conducting steel driven at a fixed velocity along the x-direction. In the first instance, mirror symmetry with respect to the $x-z$ plane and periodicity along the x-direction are assumed, which means that end effects are neglected.

This allows reducing the model to the symmetry cell depicted in Fig. 1, which contains one coil wound around a magnetic core and supplied with a DC current. The steady state problem is solved with the standard $A-V$ magnetodynamic 3D formulation, assuming $\partial_t \mathbf{a} = 0$.

Different numerical experiments will be presented in the full paper. In general, the system turns out to be very sensitive to the local mesh refinement, which acts on the behaviour of the SUPG scheme through the definition of the local Peclet number (10). Fig. 2 represents the induced current density in the mover seen from below, for an velocity v =10 km/h. As the speed inscreases, convergence becomes rapidly more difficult, which tends to indicate that the upwinding shape function can be improved. This is the purpose of the next section.

V. THE GEOMETRY OF THE CONVECTION OPERATOR

In [4], a geometrical interpretation for the discretization of the convection operator in terms of Whitney forms, the Lie derivative and the extrusion operator defined by Bossavit in [5] is presented. It is shown that this discretization introduces naturally a kind of up-winding quadrature for scalar functions, i.e. for Whitney forms of degree 0. This interpretation provides a consistent mean for generalizing the SUPG scheme to differential forms of higher degrees, and in particular to Whitney edge elements. The theoretical developments as well as numerical comparisons with the scheme of Xu will be presented in the full paper.

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