Reinterpretation of the *Nodal Force Method* within discrete geometric approaches

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***Abstract*— We propose a geometric reinterpretation of the *Nodal Force Method* in the framework of a pair of discrete formulations for magnetostatics on complementary meshes.**

***Index Terms*— Discrete approaches, Forces, Magnetostatics.**

I. Introduction

The force distribution on a body can be computed by means of the so called "Nodal Force Method" (NFM), proposed by different Authors [1], [2] in the framework of finite elements. The aim of this paper is provide a geometric reinterpretation of the NFM, when used within discrete geometric approaches. We will focus on magnetostatics, but the same interpretation holds for eddy-currents problems. We will consider a pair of discrete formulations[1] on complementary tetrahedral meshes to solve the magnetostatic problem, both in terms of the circulation of the magnetic vector potential and in terms of a scalar magnetic potential and the circulation of the electric vector potential, where needed. In both the cases, we will express the force acting on a node n of a tetrahedron v in terms of the geometric entities of the mesh and of the global electromagnetic quantities like the fluxes of the induction field or the circulations of the magnetic field.

II. Discrete formulations for magnetostatics

The domain of interest D consists of a source region D_s, where known currents are present, and of a region D_m, where magnetic materials are present; the complement of D_s and D_m in D is the insulating region D_a. We introduce in D a pair of interlocked cell complexes. One complex is made of simplexes (the 3-cells are tetrahedra), while the other is obtained from it, according to the barycentric subdivision. We denote by $\mathcal{K}$ the primal complex (whose cells are endowed with *inner* orientation) and by $\tilde{\mathcal{K}}$ the dual complex (whose cells are endowed with *outer* orientation) [3]. As the same geometric element of a complex can be thought with two complementary orientations, we may construct the pair of meshes $\mathcal{M}' = (\mathcal{K}^s, \tilde{\mathcal{K}})$ and $\mathcal{M}'' = (\mathcal{K}, \tilde{\mathcal{K}}^s)$, where the suffix "^{s}" indicates the simplicial complex. We will denote by $\mathcal{M}$ either $\mathcal{M}'$ or $\mathcal{M}''$. In addition, the interconnections between the p-cells of the primal complex of a mesh $\mathcal{M}$ are described by means of the usual incidence matrices. In particular for $\mathcal{K}^s$, we denote by $\mathbf{G}$ the incidence matrix between edges and nodes, $\mathbf{C}$ between faces and edges and $\mathbf{D}$ between volumes and faces; similarly for the dual complex $\tilde{\mathcal{K}}^s$ we write $\tilde{\mathbf{G}}$, $\tilde{\mathbf{C}}$ and $\tilde{\mathbf{D}}$ respectively.

Next, we consider the integrals of the field quantities with respect to the p-cells of mesh $\mathcal{M}$, yielding the Degrees of Freedom (DoF) arrays whose elements are indexed over the corresponding p-cells. Therefore, we obtain that $\mathbf{\Phi}$ is the DoF-array of magnetic induction fluxes associated with primal faces f, $\mathbf{F}$ is the DoF-array of m.m.f.s associated with dual edges $\tilde{e}$, $\mathbf{I}$ is the DoF-array of electric currents associated with dual faces $\tilde{f}$. In the following, we briefly recall two possible discrete formulations (in the full paper we will give all the details).

A. Formulation in $\mathcal{M}'$

In D we consider the mesh $\mathcal{M}'$. We search for array $\mathbf{A}$ of circulations A of the magnetic vector potential along the primal edges of $\mathcal{K}^s$ such that $\mathbf{\Phi} = \mathbf{C}\mathbf{A}$ and

$$\mathbf{C}^T \boldsymbol{\nu} \mathbf{C}\, \mathbf{A} = \mathbf{I} \qquad (1)$$

hold; $\mathbf{I}$ has non null entries for the dual faces of $\tilde{\mathcal{K}}$ in D_s. Matrix $\boldsymbol{\nu}$ (dim($\boldsymbol{\nu}$) $= N_f$, N_f being the number of faces in $\mathcal{K}^s$) is the reluctivity matrix such that $\mathbf{F} = \boldsymbol{\nu}\,\mathbf{\Phi}$. This matrix can be computed according to the following approaches, [4], [5], under the hypothesis of element wise uniform fields.

B. Formulation in $\mathcal{M}''$

In D we consider the mesh $\mathcal{M}''$. We search for array $\mathbf{\Omega}$ of the magnetic scalar potentials Ω associated with the dual nodes of $\tilde{\mathcal{K}}^s$ such that $\mathbf{F} = \tilde{\mathbf{G}}\mathbf{\Omega}$ and

$$\tilde{\mathbf{G}}^T \boldsymbol{\mu} \tilde{\mathbf{G}}\, \mathbf{\Omega} = -\tilde{\mathbf{G}}^T \boldsymbol{\mu}\, \mathbf{T} \qquad (2)$$

hold, where $\mathbf{T}$ is the array of circulations of the electric vector potential T along dual edges; it has non null entries for the edges of $\tilde{\mathcal{K}}^s$ in D_s. Matrix $\boldsymbol{\mu}$ (dim($\boldsymbol{\mu}$) $= N_{\tilde{e}}$, $N_{\tilde{e}}$ being the number of edges in $\tilde{\mathcal{K}}^s$) is the permeability matrix such that $\mathbf{\Phi} = \boldsymbol{\mu}\,\mathbf{F}$, [6]; elementwise uniform fields are again assumed.

III. The Nodal Force Method

We indicate with L a layer of tetrahedra enclosing the magnetic domain D_m, such that $L \subset D_a$ and each tetrahedron $v \in L$ may have 1, 2 or 3 nodes on ∂D_m; we denote by n one of these nodes and with $\mathcal{N}$ the set they form. Then the

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[1]These formulations are part of the GAME (Geometric Approach for Maxwell Equations) code developed by R. Specogna and F. Trevisan.

magnetic force F_n associated with node n of v can be written as [2], [1]

$$\mathrm{F}_n = -\int_v \sigma_m \cdot \nabla\gamma, \tag{3}$$

where $\sigma_m = (\mathrm{HB} - \frac{1}{2}\mathrm{H}\cdot\mathrm{B\,I})$ is the Maxwell stress tensor in terms of the magnetic induction field B and the magnetic field H in the vacuum, I is the identity tensor. Finally γ is an arbitrary function (we need at least to compute the gradient of it) with support in L; it is 1 on ∂D_m and 0 on $\partial L - \partial D_m$. We will concentrate on the single tetrahedron v, since he resultant force F_{D_m} acting on the body, is the sum of the contributions F_n, with $n \in \mathcal{N}$, from all $v \in L$. It is usual to express γ inside v, as the sum of the Whitney nodal functions w_n associated with $n \in \mathcal{N}$. Then, it is easy to show [5] that in the primal complex $\mathcal{K}^s$ (or equivalently in the complex $\tilde{\mathcal{K}}^s$)

$$\nabla w_n = -\frac{1}{3\,\mathrm{vol}(v)}\,\mathbf{D}_{v,n}\mathrm{f}_n, \tag{4}$$

where f_n is the area vector whose magnitude equals the area of the face f_n (opposite to node n) and that is perpendicular to f_n and pointing in a way congruent (according to the right handed screw rule) with the orientation of that face. Entry $\mathbf{D}_{v,n}$ is the incidence number between the inner orientations of v and f_n of $\mathcal{K}^s$ (similarly between $\tilde{v}$ and $\tilde{f}_n$ of $\tilde{\mathcal{K}}^s$ the incidence is $\tilde{\mathbf{D}}_{v,n}$). Finally $\mathrm{vol}(v)$ is the volume of the tetrahedron.

A. Geometric reinterpretation using the formulation in $\mathcal{M}'$

The formulation (1) allows the computation of the fluxes of the induction field on the four faces of v. Then, we will show, that a uniform induction field B in v can be obtained from

$$\mathrm{B} = \frac{1}{3\,\mathrm{vol}(v)}\sum_{i=1}^{3}\mathbf{G}_{in}\mathrm{e}_i\,\mathbf{D}_{vi}\Phi_i, \tag{5}$$

where e_i, with $i = 1,\ldots 3$, is the edge vector associated with edge e_i drawn from node n, Φ_i is the induction flux associated with face f_i having node n as vertex; face f_i pairs with e_i . Integers $\mathbf{D}_{vi}$ and $\mathbf{G}_{in}$ are incidence numbers between orientations of v, f_i and e_i, n respectively. From it and (4), will show that (3) can be rewritten as

$$\mathrm{F}'_n = -\frac{\nu_0}{9\mathrm{vol}(v)}\sum_{i,j=1}^{3}\mathbf{D}_{vi}\Phi_i\,\mathrm{u}_{ij}^n\,\mathbf{D}_{vj}\Phi_j, \tag{6}$$

where $\mathrm{u}_{ij}^n = \mathbf{G}_{jn}\mathrm{e}_j + \frac{1}{6\mathrm{vol}(v)}\mathbf{G}_{in}\mathbf{G}_{jn}\mathrm{e}_i\cdot\mathrm{e}_j\,\mathbf{D}_{vn}\mathrm{f}_n$. It should be noted that the vector u_{ij}^n contains all the geometric information in terms of the three edges e_i drawn from the common node n and the face f_n opposite to it.

B. Geometric reinterpretation using the formulation in $\mathcal{M}''$

The formulation (2) allows the computation of the m.m.f.s along the six edges of v. Then, we will show, that a uniform magnetic field H in v can be expressed as

$$\mathrm{H} = \frac{1}{3\,\mathrm{vol}(v)}\sum_{i=1}^{3}\tilde{\mathbf{D}}_{vi}\tilde{\mathrm{f}}_i\,\tilde{\mathbf{G}}_{in}F_i, \tag{7}$$

where $\tilde{\mathrm{f}}_i$, with $i = 1,\ldots 3$, is the face vector associated with dual face $\tilde{f}_i$ having node n as vertex, F_i is the m.m.f. associated with dual edge $\tilde{e}_i$ drawn from node n; again dual face $\tilde{f}_i$ pairs with $\tilde{e}_i$. From it and (4), will show that (3) can be rewritten as

$$\mathrm{F}''_n = \frac{\mu_0}{9\mathrm{vol}(v)}\sum_{i,j=1}^{3}\tilde{\mathbf{G}}_{ni}F_i\,\mathrm{v}_{ij}^n\,\tilde{\mathbf{G}}_{nj}F_j, \tag{8}$$

where $3\mathrm{vol}(v)\mathrm{v}_{ij}^n = \tilde{\mathbf{D}}_{vn}\tilde{\mathrm{f}}_n \cdot \tilde{\mathbf{D}}_{vi}\tilde{\mathrm{f}}_i\,\tilde{\mathbf{D}}_{vj}\tilde{\mathrm{f}}_j - \frac{1}{2}\tilde{\mathbf{D}}_{vi}\tilde{\mathrm{f}}_i \cdot \tilde{\mathbf{D}}_{vj}\tilde{\mathrm{f}}_j\,\tilde{\mathbf{D}}_{vn}\tilde{\mathrm{f}}_n$. Again the vector v_{ij}^n contains all the geometric information in terms of the three dual faces $\tilde{f}_i$ drawn from the common node n and the face $\tilde{f}_n$ opposite to it.

IV. Numerical Experiment and Results

We computed the resultant force acting on a cylinder ($\mu_r = 100$) close to a circular coil (400 turns, 1 A per turn); the geometry is axisymmetric (it is shown in Fig. 1 on the right) but we solved it as a 3D magnetostatic problem using the pair of complementary formulations on $\mathcal{K}^s$, $\tilde{\mathcal{K}}^s$ having 82012 tetrahedra, 14330 nodes and 96867 edges. We obtained $F'_z = 9.06\,mN$, $F''_z = 9.82\,mN$ for the axial component of the resultant force from (6) and (8) respectively. For comparison, a 2D axisymmetric analysis with the ANSYS code computed $9.42\,mN$, using about 20000 II order quadrilateral elements.

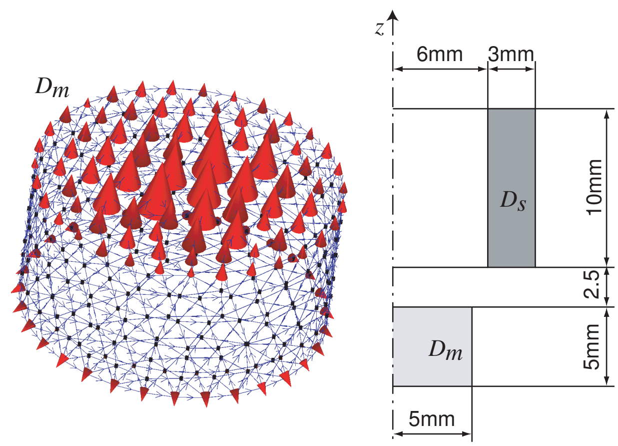

Fig. 1. On the right: geometry of the test problem. On the left: local force distribution on the boundary of the magnetic domain D_m.

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