A Theory for Electromagnetic Force Formulas in Continuous Media

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An energy-based theory for electromagnetic forces in continuous media is presented. The aim is to provide a guide and a complete toolbox for their numerical computation. In an Euclidean space, the electromechanical coupling is shown to be realized by a stress tensor, in terms of which the classical electromagnetic force formulas can be reinterpreted, unified, and generalized.

***Index Terms*—Differential geometry, electromagnetic coupling, electromagnetic forces, finite-element methods, torque.**

I. Introduction

IN MODERN computational electromagnetics, one needs more and more to compute local forces in material exhibiting saturation, anisotropy, magnetostriction, and hysteresis. The issue of forces then comes into play entangled with energy considerations. The theoretical issue has been addressed by Bossavit [1]–[4]. This paper, however, intends to provide a more operative formalism, i.e., a set of general rules and formulas that lead straightfully from the statement of the electromechanical problem to a practical implementation of a solution method for it.

The existence of such a enduring controversy about the computation of electromagnetic forces and the persistence of so many uncertainties about their implementation is certainly to ascribe to the fact that the issue cannot be completely clarified with the concepts of classical Vector and Tensor analysis. The mathematical analysis of this problem requires to consider a deforming body, and to apply energy conservation rules to it. The background required to perform such operations is differential geometry (see, e.g., [5]). Fortunately, the theoretical results can be reexpressed at the end in terms of vector and tensor fields. It turns out that the electromechanical and magnetomechanical couplings can be expressed in terms of a stress tensor. The procedure to determine that stress tensor is described in this paper.

Finally, it is shown in the last part of this paper that classical forces formulas and methods that can be found in literature and are commonly applied in numerical simulations, can be unified thanks to this coupling stress tensor. They are associated with different choices of the (possibly virtual) velocity field describing the deformation of the domain. This is not only a backwards confirmation of the proposed theory, but also a firm departure point to advisedly tackle with more complex materials.

II. Theory

A. Function

One first introduces the notation

$$u\colon x \in D \subset M \mapsto y = u(x) \in E \subset N \tag{1}$$

for a function (or a map). In this exhaustive notation, u is the name of the function, x and y are the variable and the value of the function, respectively. Finally, $D \subset M$ and $E \subset N$ are the domain and the codomain of the function. The alternative notations $D \equiv u^{-1}(E)$ and $E \equiv u(D)$ can be used if necessary. All elements of the complete definition (1) are, however, not always relevant, and shorthand notations are used whenever no confusion is possible.

B. Differential Forms

Basically, Vector and Tensor analysis distinguish scalar fields (1 component), vector fields (3 components), and tensor fields (9 components) in 3-D Euclidean spaces. Differential geometry, on the other hand, distinguishes a much larger set of fields. In particular, differential forms can be regarded as natural arguments for p-fold integrals, which amounts to say that they are associated with a map from the geometrical entities of the domain (points, curves, surfaces, and volumes) to the real numbers.

On a 3-D domain M, there exist four kinds of differential forms, called p-forms, $p = 0, 1, 2, 3$. A 0-form is a scalar function defined on M, i.e., a map from the points of M to the real numbers. Similarly, a 1-form a is a map from the curves of M to the real numbers that verifies the additivity rule

$$\int_{C_1+C_2} a = \int_{C_1} a + \int_{C_2} a \tag{2}$$

for any curves C_1 and C_2 in M. This linearity condition ensures that the map identifies one and only one vector field $\mathbf{a}$ on M such that

$$\int_C a = \int_C \mathbf{a} \cdot \mathrm{d}\mathbf{C} \tag{3}$$

holds for any curve $C \subset M$. But the two representations are nevertheless not exactly equivalent. The representation in terms of a map (1-form) still makes sense when de domain M deforms. The number associated with a given curve remains unchanged even if the curve is deformed. On the other hand, Vector analysis provides no rule to involve a vector field in the deformation of its domain of definition. This distinction is important when it comes about the definition of electromagnetic forces.

Similarly to 1-forms, 2-forms are maps from the surfaces in M onto the real numbers and 3-forms are maps from the volumes (subsets) of M onto the real numbers. They also both

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TABLE I
DIFFERENTIAL FORMS OF DEGREE 0, 1, 2, AND 3 IN A 3-D SPACE: ASSOCIATED GEOMETRICAL MAP, PHYSICAL INTERPRETATION, AND EXAMPLES ENCOUNTERED IN THIS PAPER

0–form	Points $\mapsto \mathbb{R}$	scalar function	u
1–form	Curves $\mapsto \mathbb{R}$	circulation density	$\mathbf{a}, \mathbf{e}, \mathbf{h}$
2–form	Surfaces $\mapsto \mathbb{R}$	flux density	$\mathbf{b}, \mathbf{d}, \mathbf{j}$
3–form	Volumes $\mapsto \mathbb{R}$	volume density	ρ_M^Ψ, ρ_E^Ψ

verify an additivity rule like (2). Let $\Lambda^p(M)$ denote the set of all p-forms defined on M. Table I summarizes the different kinds of differential forms and gives the associated geometrical map, their physical interpretation and the examples encountered in this paper and in [6].

C. Co-Moving Time Derivative

Thanks to the concept of p-forms, fields can be consistently defined on deforming domains. In order to establish the way they vary in time, an Eulerian representation is now adopted, i.e., the fields are defined on a subset Ω of an Euclidean space E and the deformation of Ω is described by the velocity field $\mathbf{v}$. The co-moving time derivative $\mathcal{L}\mathbf{v}$ is then defined by the property

$$\partial_t \int_{C^{(p)}} a_{(p)} = \int_{C^{(p)}} \mathcal{L}_{\mathbf{v}} a_{(p)} \tag{4}$$

where $C^{(p)}$ is any p-dimensional geometrical subset of Ω and $a_{(p)}$ any p-form defined on Ω. The co-moving time derivative determines how time derivative and integration over space commute. It allows obtaining the local form (partial differential equations) of global energy balances on deforming domains.

Differential geometry provides the expressions of the co-moving time derivatives of p-forms

$$\mathcal{L}_{\mathbf{v}} f = \dot{f} \tag{5}$$
$$\mathcal{L}_{\mathbf{v}} \mathbf{h} = \dot{\mathbf{h}} + (\nabla \mathbf{v}) \cdot \mathbf{h} \tag{6}$$
$$\mathcal{L}_{\mathbf{v}} \mathbf{b} = \dot{\mathbf{b}} - \mathbf{b} \cdot (\nabla \mathbf{v}) + \mathbf{b}\,\mathrm{tr}(\nabla \mathbf{v}) \tag{7}$$
$$\mathcal{L}_{\mathbf{v}} \rho = \dot{\rho} + \mathrm{tr}(\nabla \mathbf{v})\, \rho. \tag{8}$$

The definitions of the dot operator and of products like $\mathbf{a}\cdot(\nabla \mathbf{v})$ and $(\nabla \mathbf{v})\cdot \mathbf{a}$ are given in the Appendix.

In order to make the link with well-established mathematical notions, it should be pointed out that $\mathcal{L}_{\mathbf{v}} \equiv \partial_t + \mathcal{L}_{\mathbf{v}}$, where $\mathcal{L}_{\mathbf{v}}$ is the Lie derivative [5]. One may recognize in (5) and (8) the material derivatives encountered in Fluid dynamics. Equations (6) and (7) could, therefore, be regarded as the material derivatives of 1- and 2-forms, respectively. But as electromagnetic fields do not need material support, the name co-moving time derivative is preferred [5].

III. A WORKED-OUT EXAMPLE

We have so gathered the theoretical elements needed to analyze the magnetomechanical coupling in a continuous medium. Let us assume that the magnetic energy density of a magnetostrictive material is given by the functional

$$\rho_M^\Psi \colon \mathbf{b}, \varepsilon \mapsto \Lambda^3(\Omega) \tag{9}$$

where $\mathbf{b} = \mathrm{curl}\,\mathbf{a}$ is the induction field and ε is the strain tensor. Throughout the paper, the density of a quantity X will be denoted ρ^X.

The constitutive relations associated with the energy density functional are

$$\tilde{\mathbf{h}} = \partial_{\mathbf{b}} \rho_M^\Psi, \quad \tilde{\sigma} = \partial_\varepsilon \rho_M^\Psi \tag{10}$$

with $\tilde{\mathbf{h}}$ the magnetic field (one assumes there is no hysteresis) and $\tilde{\sigma}$ the magnetostriction stress, which is a symmetrical tensor. The variation with time of the magnetic energy contained in the system Ω is

$$\partial_t \Psi_M = \partial_t \int_\Omega \rho_M^\Psi = \int_\Omega \mathcal{L}_{\mathbf{v}} \rho_M^\Psi \tag{11}$$

by (4). Using (8) and the chain rule of derivatives to expand $\dot{\rho}_M^\Psi$, one obtains

$$\mathcal{L}_{\mathbf{v}} \rho_M^\Psi = \left\{ \left(\partial_{\mathbf{b}} \rho_M^\Psi\right) \cdot \dot{\mathbf{b}} + \left(\partial_\varepsilon \rho_M^\Psi\right) : \dot{\varepsilon} \right\} + \mathrm{tr}(\nabla \mathbf{v})\, \rho_M^\Psi$$

with the tensor product $a{:}\, b = a_{ij} b_{ij}$. Using now (10) and (7) to substitute for $\dot{\mathbf{b}}$ yields

$$\begin{aligned} \mathcal{L}_{\mathbf{v}} \rho_M^\Psi &= \tilde{\mathbf{h}} \cdot \mathcal{L}_{\mathbf{v}} \mathbf{b} + \mathbf{b} \cdot \nabla \mathbf{v} \cdot \tilde{\mathbf{h}} + \tilde{\sigma} {:}\, \nabla \mathbf{v} \\ &\quad - \mathrm{tr}(\nabla \mathbf{v}) \{ \tilde{\mathbf{h}} \cdot \mathbf{b} - \rho_M^\Psi \} \end{aligned} \tag{12}$$

where we have also used the fact that $\dot{\varepsilon}$ is the symmetrical part of $\nabla \mathbf{v}$. As explained in more detail in [6], the mechanical work delivered by magnetic forces is

$$\dot{W}_M \equiv -\, \partial_t \Psi_M |_{\mathcal{L}_{\mathbf{v}} \mathbf{b} = 0} = - \int_\Omega \sigma_{\mathrm{em}} {:}\, \nabla \mathbf{v}. \tag{13}$$

The zero co-moving time derivative of $\mathbf{b}$ in (13) is the precise mathematical statement of what is commonly formulated *holding the fluxes constant*. Substituting (12) in (11), and factorizing $\nabla \mathbf{v}$, the Maxwell stress tensor of the magnetostrictive material is obtained

$$\sigma_{\mathrm{em}} = \mathbf{b}\tilde{\mathbf{h}} + \tilde{\sigma} - \{ \tilde{\mathbf{h}} \cdot \mathbf{b} - \rho_M^\Psi \} \mathbb{I} \tag{14}$$

where $\mathbb{I}$ is the identity matrix. Note the use of the dyadic (undotted) vector product $(\mathbf{v}\mathbf{w})_{ij} = v^i w^j$.

Besides the magnetostriction stress $\tilde{\sigma}$, which is a material property, one has extra terms in (14) that account for the so-called form effect. An example of a magnetostrictive energy functional like (9) is proposed in [7].

IV. COMPUTATION OF ELECTROMAGNETIC FORCES

In the last part of this paper, the implication of the results of Section III on the definition of electromagnetic forces are reviewed. It is shown, in particular, that the classical electromagnetic force formulas used in finite-element computations can be derived from (13) by considering different velocity fields $\mathbf{v}$.

1) Maxwell Stress: In its most fundamental expression, the magnetomechanical and electromechanical couplings is the product of a stress tensor with the gradient of the velocity field, and *not* the product of a force density with the velocity field.

This is due to the fact that the geometry dependent terms in (5)–(8) involve $\nabla \mathbf{v}$, but *not* $\mathbf{v}$ itself.

2) Energy Density: Each material has its own Maxwell stress tensor σ_{em}, whose expression is obtained from the energy density ρ_M^Ψ by applying the same procedure as before. For instance, the Maxwell stress tensor of a material that has at the same time a magnetic energy density $\rho_M^\Psi\colon \mathbf{b} \mapsto \Lambda^3(\Omega)$ and an electric energy density $\rho_E^\Psi\colon \mathbf{d} \mapsto \Lambda^3(\Omega)$, is defined by

$$\partial_t \Psi_M|_{\mathcal{L}_\mathbf{v}\mathbf{b}=0} + \partial_t \Psi_E|_{\mathcal{L}_\mathbf{v}\mathbf{d}=0} = \int_\Omega \sigma_{\text{em}}\colon \nabla \mathbf{v}. \quad (15)$$

One finds

$$\sigma_{\text{em}} = \mathbf{d}\tilde{\mathbf{e}} + \mathbf{b}\tilde{\mathbf{h}} - \left\{ \tilde{\mathbf{e}} \cdot \mathbf{d} + \tilde{\mathbf{h}} \cdot \mathbf{b} - \rho_M^\Psi(\mathbf{b}) - \rho_E^\Psi(\mathbf{d}) \right\} \mathbb{I}. \quad (16)$$

Other examples of materials can be found in [8].

3) Force Density: The link between the Maxwell stress tensor σ_{em} and the electromagnetic force density is found by integrating (13) by part over Ω. One has

$$\int_\Omega \sigma_{\text{em}}\colon \nabla \mathbf{v} = -\int_\Omega \rho_{\text{em}}^{\mathbf{f}} \cdot \mathbf{v} + \int_{\partial\Omega} \mathbf{n} \cdot \sigma_{\text{em}} \cdot \mathbf{v} \quad (17)$$

with $\rho_{\text{em}}^{\mathbf{f}} \equiv \text{div}\sigma_{\text{em}}$ by definition and $\mathbf{n}$ the exterior normal to $\partial\Omega$. One can see the electromagnetic forces have a volume component $\rho_{\text{em}}^{\mathbf{f}}$ *and* a surface component $\mathbf{n} \cdot \sigma_{\text{em}}$.

4) Continuity: At material interfaces, the Maxwell stress tensor is, in general, discontinuous. The force density is then defined, in the sense of distributions, as the jump of $\sigma_{\text{em}} \cdot \mathbf{n}$. This can be seen by applying (17) material domain by material domain and summing up all contributions.

5) Sign Convention: It should be noted that the Maxwell stress tensor σ_{em} is a true mechanical stress, i.e., its work is delivered by the mechanical energy compartment and received by the electromagnetic compartment. On the other hand, $\rho_{\text{em}}^{\mathbf{f}}$ is a magnetic force. The work it delivers is withdrawn from the electromagnetic compartment and received by the mechanical compartment.

6) Applied Stress: The Maxwell stress tensor can be used directly as an applied stress in the structural equations and boundary conditions of the system. One has

$$\text{div}\left\{\sigma + \sigma_{\text{em}}\right\} + \rho^{\mathbf{f}} = 0 \quad (18)$$

which is easier than coupling through the forces

$$\text{div}\sigma + \left\{\rho^{\mathbf{f}} + \rho_{\text{em}}^{\mathbf{f}}\right\} = 0 \quad (19)$$

since $\rho_{\text{em}}^{\mathbf{f}}$ requires a special treatment at material interfaces.

7) Force-Free Region Z: Applying (17) to a force-free region Z, i.e., $\rho_{\text{em}}^{\mathbf{f}} = 0$ on Z, yields

$$\int_Z \sigma_{\text{em}}\colon \nabla \mathbf{v} = \int_{\partial Z} \mathbf{n} \cdot \sigma_{\text{em}} \cdot \mathbf{v}. \quad (20)$$

This shows that the velocity field $\mathbf{v}$ is arbitrary on the *interior* of a force-free region. If the force-free region Z surrounds a moving region Y (Fig. 1), $\mathbf{v}$ cannot, however, be set to zero everywhere in Z. A transition region $S \subset Z$ must be present to preserve the continuity of $\mathbf{v}$, which is different from zero on δY.

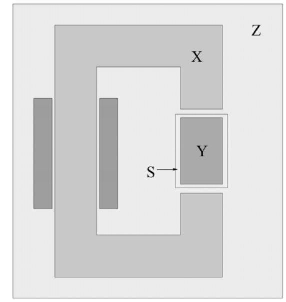

Fig. 1. Typical resultant magnetic force problem, Y is the moving rigid region (body), Z is the force-free region, X is fixed, and $S \subset Z$ is the eggshell.

8) Rigid Region Y: Considering a moving rigid body Y (Fig. 1) whose velocity field $\mathbf{v} = \mathbf{v}_0 + \mathbf{w}_0 \times \mathbf{r}$, (17) gives

$$\int_Y \hat{\sigma}_{\text{em}} \cdot \mathbf{w}_0 = -\int_Y \rho_{\text{em}}^{\mathbf{f}} \cdot \{\mathbf{v}_0 + \mathbf{w}_0 \times \mathbf{r}\} + \int_{\partial Y} \mathbf{n} \cdot \sigma_{\text{em}} \cdot \{\mathbf{v}_0 + \mathbf{w}_0 \times \mathbf{r}\} \quad (21)$$

with $\hat{\sigma}_{\text{em}}^k = \epsilon^{ijk}(\sigma_{\text{em}})_{ij}$, where ϵ^{ijk} is the Levi–Civita symbol (see, e.g., [5]). The vectors $\mathbf{v}_0$ and $\mathbf{w}_0$ being arbitrary and constant on Y, one may factorize them to define the resultant magnetic force

$$\mathbf{F}_Y = \int_Y \rho_{\text{em}}^{\mathbf{f}} = \int_{\partial Y} \mathbf{n} \cdot \sigma_{\text{em}} \quad (22)$$

and the resultant magnetic torque

$$\mathbf{T}_Y = \int_Y \left\{\hat{\sigma}_{\text{em}} + \mathbf{r} \times \rho_{\text{em}}^{\mathbf{f}}\right\} = \int_{\partial Y} \mathbf{r} \times (\mathbf{n} \cdot \sigma_{\text{em}}) \quad (23)$$

acting on the rigid region Y. The term $\hat{\sigma}_{\text{em}}$ is zero when σ_{em} is symmetric.

Equations (22) and (23) show that the resultant force $\mathbf{F}_Y$ and the resultant torque $\mathbf{T}_Y$ acting on a rigid region Y can both be evaluated by means of a the surface integral on its boundary ∂Y of the Maxwell stress tensor of empty space. This classical result is known as the Maxwell stress tensor method. Note that the rigid region needs not be identified with a material body. It may be larger, provided that the extra domain enclosed is force free.

9) Eggshell Method: In practice, it is easier to work with volume integrations which are already implemented in the finite-element program. In order to avoid the surface integration in (22) and (23), which requires a specific implementation, one chooses a domain $\Omega = Y + S$ (Fig. 1) larger than the rigid region Y, i.e., enclosing as well a part of a force-free region S (generally air). One defines on Ω a velocity field which describes a rigid motion of Y, decays smoothly in S, and vanishes on $\partial\Omega$, i.e., $\mathbf{v} = \{\mathbf{v}_0 + \mathbf{w}_0 \times \mathbf{r}\}\gamma$, where γ is any smooth function whose value is 1 on Y and 0 on $\partial\Omega$. Applying now (20) to the force-free region $\Omega - Y$ yields successively

$$\begin{aligned} -\int_{\Omega - Y} \sigma_{\text{em}}\colon \nabla \mathbf{v} &= -\int_{\partial\Omega - \partial Y} \mathbf{n} \cdot \sigma_{\text{em}} \cdot \mathbf{v} \\ &= \int_{\partial Y} \mathbf{n} \cdot \sigma_{\text{em}} \cdot \{\mathbf{v}_0 + \mathbf{w}_0 \times \mathbf{r}\} \\ &= \mathbf{F}_Y \cdot \mathbf{v}_0 + \mathbf{T}_Y \cdot \mathbf{w}_0 \end{aligned} \quad (24)$$

by the definition of γ, (22) and (23). This gives an alternative way to compute the resultants $\mathbf{F}_Y$ and $\mathbf{T}_Y$, now by means of a volume integral, with $\sigma_{\rm em}$ the Maxwell stress tensor of empty space. In the particular case of a translation velocity field $\mathbf{v} = \mathbf{v}_0\gamma$, one has

$$\mathbf{F}_Y = -\int_S \sigma_{\rm em} \cdot \nabla\gamma. \tag{25}$$

In practice, the shell region S is reduced to a minimum. The shell can be defined explicitly by the user, like in Fig. 1, and γ is then also a user-defined analytic function. In general, it is easier to have the eggshell automatically defined. A natural choice is to take one layer of finite elements around the moving region, i.e., all elements touching the boundary ∂Y on the outer side. The function γ is then simply, on that support, the sum of the shape functions associated with the nodes of Y. This alternative method to compute the resultant force on rigid bodies by means of a volume integration instead of a surface integration has been validated in [9] and [10].

10) Arkkio's Method: The torque in 2-D models of electrical rotating machines can be calculated with the eggshell method by considering the rotation velocity field

$$\mathbf{v} = \gamma \mathbf{w}_0 \times \mathbf{r} = \frac{R_o - r}{R_o - R_i}\{w_0 \mathbf{e}_z\} \times \{r\mathbf{e}_r\} \tag{26}$$

in cylindrical coordinates, where R_o and R_i are, respectively, the outer and inner radius of any cylindrical air region S contained in the air gap. The gradient of the velocity field is

$$\nabla\mathbf{v} = \begin{pmatrix} \partial_r v^r & \frac{1}{r}\partial_\theta v^r \\ r\partial_r(\frac{v^\theta}{r}) & \partial_\theta(\frac{v^\theta}{r}) \end{pmatrix} = \frac{-w_0 r}{R_o - R_i}\mathbf{e}_r\mathbf{e}_\theta \tag{27}$$

when by (24), the formula of Arkkio [11]

$$\mathbf{T}_Y = \frac{\mathbf{e}_z}{R_o - R_i}\int_S r(\sigma_{\rm em})_{r\theta}. \tag{28}$$

11) Coulomb's Method: Coulomb's formula to compute nodal electromagnetic forces by the local derivative of the Jacobian [12], is obtained by identifying $\mathbf{v}_0$ with the virtual velocity of one node and the function γ with the shape function of that node, $\mathbf{v} = \mathbf{v}_0\gamma$. One obtains

$$\{-\int_\Omega \sigma_{\rm em}\colon \nabla\gamma\} \cdot \mathbf{v}_0 = \left\{\int_\Omega \rho^{\mathbf{f}}_{\rm em}\gamma\right\} \cdot \mathbf{v}_0 \tag{29}$$

where the domain of integration can be limited to the support of the nodal shape function γ. This allows to define the nodal net force $\mathbf{F}_N$ acting on the node by

$$\mathbf{F}_N \equiv \int_\Omega \rho^{\mathbf{f}}_{\rm em}\gamma = -\int_\Omega \sigma_{\rm em}\colon \nabla\gamma \tag{30}$$

which is identical to the formula proposed in [13]. It can also be shown to be equivalent to the formulas presented in [14] for linear materials and in [12] and [15] for nonlinear materials. Equation (30) is, however, more general, for it does not assume a particular form of $\sigma_{\rm em}$ and does not rely on a finite-element mesh. The nodal shape function is here just used as a convenient representation of a local deformation. The trick of the derivative of the jacobian is here accounted for by the co-moving time derivative.

V. Conclusion

We have shown that the electro- and magnetomechanical couplings are realized by a stress tensor, for which the name Maxwell stress tensor has been retained. This tensor has a local meaning. Each material has its own Maxwell stress tensor and it has been shown how it can be derived from a known expression of the magnetic and electric energy densities of the material. The complete mathematical developments of this theory involve Differential geometry concepts but an operative Tensor/Vector analysis formalism is obtained by supplementing the standard theory with a very limited number of new notions and formulas. Classical force fomulas are unified and generalized in this respect.

Appendix

$$(\nabla\mathbf{G}) \cdot \mathbf{F} = \sum_{ij} \mathbf{e}^i \frac{\partial G^j}{\partial x^i} F^j$$

$$\mathbf{F} \cdot (\nabla\mathbf{G}) = \sum_{ij} F^i \frac{\partial G^j}{\partial x^i} \mathbf{e}^j$$

$$\mathbf{F} \times \mathrm{curl}\mathbf{G} = (\nabla\mathbf{G}) \cdot \mathbf{F} - \mathbf{F} \cdot (\nabla\mathbf{G})$$

$$\dot{z} = \partial_t z + \mathbf{v} \cdot \mathrm{grad} z$$

$$\dot{\mathbf{z}} = \partial_t \mathbf{z} + \mathbf{v} \cdot (\nabla\mathbf{z}).$$

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