Fluxes, Inductances and their numerical computation

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***Abstract*— This paper proposes a theoretical definition of the notion of inductance valid in static and dynamic cases, and in the presence of motion or magnetic hysteresis. Different methods to compute inductances with the finite element method are compared. Their accuracy in function of frequency and mesh refinement is discussed in detail. Comparison of different kind of meshes (structured, unstructured) and of different mesh generators (ANSYS, gmsh) is presented as well.**

I. Magnetic energy balance

Considering Magnetism from the point of view of energy, the state variable is the vector potential $\mathbf{a}$ and the magnetic energy of any system Ω is a functional

$$\Psi_M \equiv \int_\Omega \rho_M^\Psi \, : \mathrm{curl}\, \mathbf{a} \mapsto \mathbb{R}. \tag{1}$$

From the mathematical point of view, the variation of magnetic energy can be expressed by the chain rule of derivative as

$$\partial_t \Psi_M = \int_\Omega \mathbf{h}_r \cdot \mathcal{L}_\mathbf{v} \, \mathrm{curl}\, \mathbf{a} + \int_\Omega \left\{ \mathcal{L}_\mathbf{v} \, \rho_M^\Psi \right\} (\mathrm{curl}\, \mathbf{a}). \tag{2}$$

where the co-moving time derivative $\mathcal{L}_\mathbf{v}$ is a time derivative that accounts for a possible motion or deformation of the domain Ω [1]. In the absence of motion, $\mathbf{v} \equiv 0$ yields $\mathcal{L}_\mathbf{v} \equiv \partial_t$ (See also [2], [3]). In (2), $\mathbf{h}_r \equiv \partial_\mathbf{b} \rho_M^\Psi$ is the reversible part (it derives from a potential, which is the magnetic energy density ρ_M^Ψ) of the magnetic field, which accounts for the magnetization phenomenon, i.e. the alignment of microscopic magnetic moments. The second terms in (2) accounts for the variation of Ψ_M when deforming Ω holding $\mathbf{a}$, i.e. the magnetic fluxes, constant.

From the point of view of Thermodynamics now, the balance of magnetic energy writes

$$\partial_t \Psi_M = \dot{W} + \dot{Q} + \dot{W}_M \tag{3}$$

for arbitrary evolutions of the system, i.e. $\forall \mathcal{L}_\mathbf{v}\, \mathbf{a}$. The rate of magnetic work

$$\dot{W} = \int_\Omega \{ \mathbf{j} + \mathcal{L}_\mathbf{v}\, \mathbf{d} \} \cdot \mathcal{L}_\mathbf{v}\, \mathbf{a} - \int_{\partial\Omega} \mathbf{n} \times \mathbf{h}_\partial \cdot \mathcal{L}_\mathbf{v}\, \mathbf{a} \tag{4}$$

is the power delivered by the work of generalised forces on the variation of the state variable $\mathbf{a}$. It decomposes into a volume term for which the generalised force is the total current density (including displacement currents) $\mathbf{j} + \mathcal{L}_\mathbf{v}\, \mathbf{d}$, and a surface term for which the generalised force is the given surface (tangent) magnetic field $\mathbf{n} \times \mathbf{h}_\partial$, and which stands for the effect on Ω of currents flowing outside Ω. The dissipation functional writes

$$\dot{Q} = -\int_\Omega \mathbf{h}_i \cdot \mathrm{curl}\, \mathcal{L}_\mathbf{v}\, \mathbf{a} \leq 0 \tag{5}$$

where $\mathbf{h}_i$ is the irreversible part of the magnetic field, which accounts for the local dissipation due to magnetic hysteresis. Finally, $\dot{W}_M$ represents the magnetic energy converted into mechanical energy, i.e. the power delivered by magnetic forces.

Identifying both definitions of $\partial_t \Psi_M$ and factorising the arbitrary $\mathcal{L}_\mathbf{v}\, \mathbf{a}$, one obtains Ampere's law $\mathrm{curl}\, \{ \mathbf{h}_r + \mathbf{h}_i, \} = \mathbf{j} + \mathcal{L}_\mathbf{v}\, \mathbf{d}$ as Euler-Lagrange equation whereas the remaining two terms must sum up to zero separately, which defines the power delivered by magnetic forces

$$\dot{W}_M = \int_\Omega \left\{ \mathcal{L}_\mathbf{v} \, \rho_M^\Psi \right\} (\mathrm{curl}\, \mathbf{a}). \tag{6}$$

II. Definition of inductance

An arbitrary system with a conductor C (coil or massive) carrying a given current I, is now considered. Whereas the field magnetic state variable is the vector potential $\mathbf{a}$, the idea behind the notion of inductance is to work with a scalar state variable. This state variable is the flux φ embraced by the conductor. In order to relate the two representations, one needs a definition of φ in function of $\mathbf{a}$, i.e. a mapping $\varphi : \mathbf{a} \mapsto \mathbb{R}$. For the sake of clarity, the simplest case is first considered. One assumes $\mathbf{d} \equiv 0$ (no coupling with electric energy), $\mathbf{h}_\partial \equiv 0$ (no coupling with external currents), $\mathbf{h}_i \equiv 0$ (no hysteresis) and $\mathbf{v} \equiv 0, \dot{W}_M \equiv 0$ (no deformation), so that (3) simplifies into

$$\partial_t \Psi_M = \int_\Omega \mathbf{j} \cdot \partial_t \mathbf{a}. \tag{7}$$

The sought relation between $\mathbf{a}$ and φ arises from the banal observation that the current density j can be written $\mathbf{j} = I\mathbf{w}$, where the current shape function $\mathbf{w}$ has support on the conducting region $C \subset \Omega$. Note that this entails no approximation if $\mathbf{w}$ is allowed to depend on time. If it is now required that magnetic work is exactly represented, i.e.

$$\int_\Omega \mathbf{j} \cdot \partial_t \mathbf{a} \equiv I \; \partial_t \varphi \quad \Rightarrow \quad \partial_t \varphi = \int_\Omega \mathbf{w} \cdot \partial_t \mathbf{a}, \tag{8}$$

a mapping between the time derivatives $\partial_t \varphi$ and $\partial_t \mathbf{a}$ is obtained. In order to get a mapping between φ and $\mathbf{a}$, one makes the *assumption* that the current shape function $\mathbf{w}$ does not depend on time, so that one has

$$\varphi(\mathbf{a}) = \int_C \mathbf{w} \cdot \mathbf{a}. \tag{9}$$

The assumption on $\mathbf{w}$ is always fulfilled for coils, but it can also be fulfilled in a more restrictive way, e.g. for a given frequency in time-harmonic problems, or on a limited time interval for a linearised model.

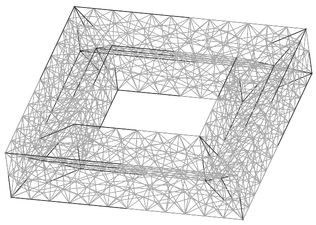

Fig. 1. Geometry and one typical mesh of the coil.

Inductance is defined by identification of magnetic energy. Two ways of doing this are possible. For a global identification, the inductance is defined as the multiplicative factor that allows writing

$$\Psi_M(\mathrm{curl}\,\mathbf{a}) = \int_0^{\varphi(\mathbf{a})} L^{-1}x \ \mathrm{d}x \quad \Rightarrow \quad L^{-1}\varphi = I. \qquad (10)$$

L is in this case *a priori* a non-linear function of all states variables of the system, and of φ in particular. In some situations, it is useful to identify the tangential inductance of the linearised system $L^{-\partial} = \partial^2_\varphi \Psi_M(\varphi^*)$ $(L^{-\partial}\partial_t\varphi = \partial_t I)$, which is a constant, but valid only in a neighborhood of φ^*.

III. Methods

Let C be the coil region depicted at Fig. 1 and Γ be the locus of the gravity centers all orthogonal cross sections of the coil, i.e. the central fiber of the coil. Let Σ be the plane square surface supported by Γ, i.e $\partial\Sigma = \Gamma$, where ∂ denotes the boundary operator. Let finally $\Sigma^- \subset \Sigma$ be the intersection of Σ with the air region surrounding C. This is a smaller square surface, placed in the air and closing the aperture of the coil. Three methods to compute the flux φ are compared. The first method (denoted $\mathbf{b}$) consists in computing the flux of $\mathbf{b}$ through Σ, whereas the second method (denoted $\mathbf{b}-$) computes the the flux of $\mathbf{b}$ through Σ^-. The third method consists in applying (9) ; two different current shape functions $\mathbf{w}$ have been considered in this case. The one denoted I assumes C is a massive conductor, whereas the other one, denoted J, assumes C is a coil, for which $\mathbf{j}$ can be expressed analytically. This problem is linear.

IV. Calculations

Fig. 2 shows L as a function of the number of elements. The three methods lead to different values of L, the difference is up to 50%. Method $\mathbf{b}$ leaves indeed to the user the task of choosing the integration surface. Different choices lead to different values of the computed inductance. Method $\mathbf{a}$, on the contrary, has no free parameter. The integration surface is implicitly determined by the energy criteria.

Fig. 4 shows now specifically static inductances computed with method $\mathbf{a}$ with 3 different meshes (gmsh, ANSYS and gmsh with a structured grid in C), Fig. 3. One sees that for unstructured meshes, gmsh and ANSYS are equivalent. One sees also that the current density has an effect on the computed inductance. For the same geometry, the inductance of a coil (The factor n^2 has not been added here.) is in this case 10% larger than the one of a massive conductor.

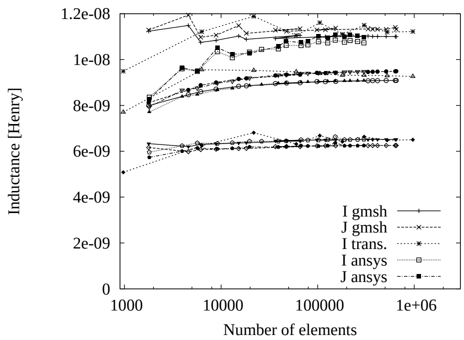

Fig. 2. Inductance in function of the number of elements. The three groups are computed with the methods $\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}^-$ resp., from top to bottom.

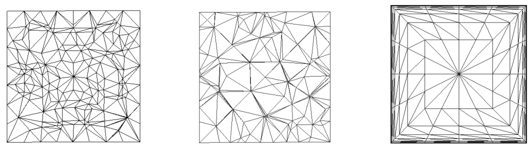

Fig. 3. Cross sections of the meshes in C with, from left to right : ANSYS, gmsh unstructured and gmsh structured.

V. Conclusions

A general definition of the notions of fluxes and inductances based on an energy balance has been proposed. The analysis has been done for a system with one single coil, but it can be generalised straightforwardly. This approach has no free parameter. The only assumption is that the current shape function does not vary too much in time. The way the definition of inductance must be adapted in the presence of couplings (hysteresis, deformation, . . .), as well as the non-linear and dynamic cases, will be addressed in the full paper.

References

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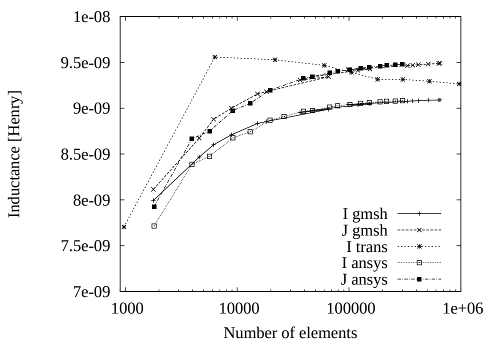

Fig. 4. Inductance as a function of mesh refinement, with 3 different meshes (gmsh, ANSYS and gmsh with a structured grid in C).