The Structure of Electromagnetic Energy Flows in Continuous Media

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A formulation of electromagnetism in continuous media is proposed that relies on the identification of the different existing electromagnetic energy reservoirs and of the flows between them. A structure is so revealed, which constitutes a natural framework to establish the partial differential equations ruling electromagnetic systems. This energy-based formulation, which unlike Maxwell's equations also integrates the material aspects, clarifies several issues related to dissipative and coupled phenomena in magnetic materials, dielectrics and conductors.

***Index Terms*—Capacitive energy storage, dielectric hysteresis, electromagnetic forces, energy conservation, inductive energy storage, magnetic hysteresis, superconducting magnet energy storage.**

I. Introduction

MAXWELL's equations are generally presented as the fundamental set of equations ruling electromagnetic (EM) phenomena. However they address only a part of the question. They do not provide any energy conservation rule and leave all material aspects aside. Consequently, they need to be complemented by constitutive laws, which are often regarded as *ad hoc* relations to close the system, not subjected to any theoretical constraint. However, in order to tackle consistently with multiphysics problems, a theory of electromagnetism with energy aspects involved from the beginning is needed.

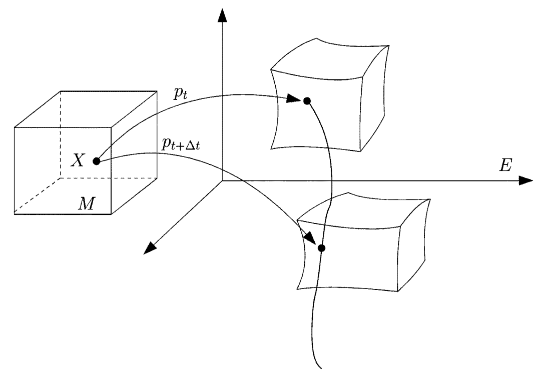

Fig. 1. Placement map at two instants of time and the trajectory of X in E.

II. Theoretical Setup

The theoretical framework needed for this purpose relies upon two manifolds: the material manifold M of which each point is associated with a material particle of the continuous medium (e.g., an atom), and the Euclidean space E which represents the space where the motion takes place and which is a manifold where a metric (i.e., the notion of distance) has been defined.

In order to describe a possible movement and/or a deformation of the system, the placement map $p_t : X \in M \mapsto x = p_t X \in E$ is defined. It associates a position in E to each material particle $X \in M$ at all instants of time $t \in [t_A, t_B]$. The codomain of the placement map, $\Omega = p_t M$, is the deformed state. On the other hand, the codomain of the map $t \in [t_A, t_B] \mapsto x = p_t X \in E$ is the trajectory of a particular material particle X (Fig. 1). The velocity field, $\mathbf{v} = \partial_t x$ (vectors in E are denoted with a bold letter), is the field of tangent vectors to all trajectories of the flow at a given instant of time.

The placement p_t is assumed to be regular and invertible at all t. It induces a 1-1 mapping, also noted p_t, of all vector and tensor fields defined on M to the corresponding vector and tensor fields defined on E. Quantities defined on M are denoted with an uppercase symbol, and corresponding quantities defined on E are denoted with a lowercase, one has $p_t Z = z$.

The classical theory of electromagnetism is expressed in terms of the so called electromagnetic fields, i.e., the vector fields $\mathbf{h}, \mathbf{b}, \mathbf{e}, \mathbf{d}, \mathbf{j}$ and the charge density ρ^Q, all defined on E. However, recent developments [1] have shown that the electromagnetic fields are more adequately defined as mappings of the material curves or surfaces (infinitesimal or not) onto real numbers, i.e., as differential forms on the material manifold M.

The set of fields selected as state variables for this energy-based approach is different. It consists of the electric scalar potential U (0-form), the magnetic vector potential A (1-form), the electric displacement D (2-form) and the current density J (2-form), all defined on M.

The metric on E attributes an intensity to the fields defined in M, thanks to p_t. For instance, the magnetic flux density writes $B \equiv \mathrm{d}A$ in M, since the exterior derivative d is the differential geometry equivalent of the curl operator in E, i.e., $p_t \mathrm{d}A = \mathrm{curl}\mathbf{a}$. The magnetic flux density associates a flux φ (in Weber) to any infinitesimal material surface Σ in M. But one needs the measure of $p_t\Sigma$ in E, and hence the metric on E, to determine the local intensity of the field $\varphi/\mathrm{measure}(p_t\Sigma)$. The magnetic energy density is thus a function of $\mathrm{curl}\mathbf{a}$ (not of $\mathrm{d}A$), and of possible other arguments like temperature, strain, etc., represented by x. If the magnetic energy is noted Ψ_M and its corre-

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sponding density ρ_M^Ψ (the density of any quantity X is denoted ρ^X), one has

$$\begin{aligned}\Psi_M(\mathrm{curl}\mathbf{a},x) &\equiv \int_{\Omega=p_tM} \rho_M^\Psi(p_t\mathrm{d}A,x)\\ &= \int_M (p_t^{-1}\rho_M^\Psi)(\mathrm{d}A,x)\\ &\equiv \left(p_t^{-1}\Psi_M\right)(\mathrm{d}A,x).\end{aligned}$$

One sees that the placement p_t (Here the inverse of the map, but invertibility is assumed.) suffices to define the expression of the energy density functional in M corresponding to the one given in E. (At least in theory. In practice, the mapping can be technically difficult.) Identical considerations apply to the mapping of all functionals introduced in this paper.

The time derivatives of the functionals require a special care. With the commutation properties $\partial_t \int_M = \int_M \partial_t, \partial_X p_t^{-1} = p_t^{-1}\partial_x$ and $\partial_t p_t^{-1} = p_t^{-1}\mathcal{L}_\mathbf{v}$, where $\mathcal{L}_\mathbf{v}$ stands for the material derivative and $\mathbf{v}$ for the velocity field associated with the placement p_t, one has successively

$$\begin{aligned}\partial_t\Psi(x) &= \partial_t \int_\Omega \rho^\Psi(x)\\ &= \partial_t \int_{M=p_t^{-1}\Omega} \left(p_t^{-1}\rho^\Psi\right)\left(p_t^{-1}x\right)\\ &= \int_M \partial_t \left\{\left(p_t^{-1}\rho^\Psi\right)(X)\right\}\\ &= \int_M \left(\partial_X p_t^{-1}\rho^\Psi\right)(X)\wedge\partial_t X\\ &\quad + \int_M \left(\partial_t p_t^{-1}\rho^\Psi\right)(X)\\ &= \int_M \left(p_t^{-1}\partial_x\rho^\Psi\right)\left(p_t^{-1}x\right)\wedge\partial_t p_t^{-1}x\\ &\quad + \int_M \left(\partial_t p_t^{-1}\rho^\Psi\right)\left(p_t^{-1}x\right)\\ &= \int_M \left(p_t^{-1}\partial_x\rho^\Psi\right)\left(p_t^{-1}x\right)\wedge p_t^{-1}\mathcal{L}_\mathbf{v}x\\ &\quad + \int_M \left(p_t^{-1}\mathcal{L}_\mathbf{v}\rho^\Psi\right)\left(p_t^{-1}x\right)\\ &= \int_\Omega \left(\partial_x\rho^\Psi\right)(x)\cdot\mathcal{L}_\mathbf{v}x + \int_\Omega \left(\mathcal{L}_\mathbf{v}\rho^\Psi\right)(x)\end{aligned}$$

The algebraic expression in E of the material derivative of the different kinds of scalar or vector fields is

$$\mathcal{L}_\mathbf{v} f = \partial_t f + \mathbf{v}\cdot(\mathrm{grad} f) \tag{1}$$
$$\mathcal{L}_\mathbf{v}\mathbf{a} = \partial_t\mathbf{a} + \mathrm{grad}\,(\mathbf{a}\cdot\mathbf{v}) - \mathbf{v}\times\mathrm{curl}\,\mathbf{a} \tag{2}$$
$$\mathcal{L}_\mathbf{v}\mathbf{d} = \partial_t\mathbf{d} + \mathrm{curl}\,(\mathbf{d}\times\mathbf{v}) + \mathbf{v}\,\mathrm{div}\,\mathbf{d} \tag{3}$$
$$\mathcal{L}_\mathbf{v}\rho = \partial_t\rho + \mathrm{div}\,(\rho\mathbf{v}). \tag{4}$$

Whereas the scalar material derivatives for 0-forms (1) and 3-forms (4) are well-known in fluid dynamics, where they are valid for the components of the velocity field and of the momentum density, the material derivative of 1-form (2) and 2-form (3) are seldom reported in the literature [5]. Note that in the absence of motion, $\mathbf{v}\equiv 0$ and $\mathcal{L}_\mathbf{v}\equiv\partial_t$.

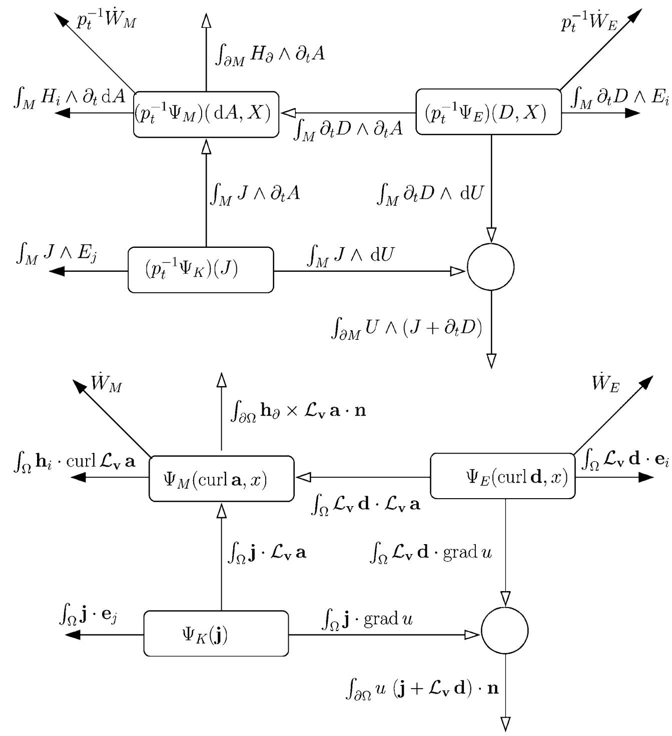

Fig. 2. EM energy flow diagram in the material manifold M (above) and in the Euclidean space E (below).

III. EM Energy Flow Diagram

Let us state at once the topology of the electromagnetic energy flow diagram as being the one depicted in Fig. 2, respectively in the material manifold M (above) or in the Euclidean space (below). Both representations are equivalent, i.e., image of one another by the placement map p_t. The following of the paper justifies this structure.

The diagram consists of four energy reservoirs, at the corners of a square. Each reservoir is associated with a particular field, resp. A, D, J, U from the upper left to the lower right corner. The A-reservoir and the D-reservoir contain respectively the magnetic and the electric energy. The U-reservoir is always empty. The J-reservoir finally, contains the kinetic energy of the charge carriers. If m_c denote the mass of one charge carrier, q_c its charge, ρ_c the density of charge carriers and $\mathbf{v}_c$ their velocity field in M, the current density in E is $\mathbf{j} = q_c\rho_c(\mathbf{v}+\mathbf{v}_c)$, and the kinetic energy density writes $\rho_K^\Psi(\mathbf{j}) = \rho_c m_c(\mathbf{v}+\mathbf{v}_c)^2/2 = \alpha\mathbf{j}^2/2$ in E, with $\alpha = m_c/(\rho_c q_c^2)$.

The flows can be classified into four categories. The white-headed arrows represent 4 internal volume flows depending on the state variables only, and 2 surface flows depending on the state variables and on a user-defined surface magnetic field $\mathbf{h}_\partial$. The black-headed arrows represent 3 dissipative volume flows in terms of the state variables (U excepted) and user-defined dissipative generalized forces $\mathbf{h}_i, \mathbf{e}_i$ and $\mathbf{e}_j$. Finally, 2 flows that are independent of the state variables connect the diagram with other energy reservoirs, in particular the mechanical one if the supplementary parameters x are held constant.

Note that the topology of the diagram and the mathematical expression of the flows are the fundamental assumptions of this formulation of Electromagnetism. They cannot be altered and make up the framework wherein any continuous medium electromagnetic system, also dissipative or coupled, inscribes. It is indeed shown further that the conservation equations associated with this fixed structure contain Maxwell's equations.

IV. Conservation Equations

As the fields A, D, J, and U are independent of each other, they can be varied freely in time in order to obtain, by a simple variational argument, the conservation equations implied by the structure of the diagram. By expressing on the one hand energy conservation at node A in M

$$\begin{aligned}\partial_t\left(p_t^{-1}\Psi_M\right)(\mathrm{d}A, X) = &\int_M (J+\partial_t D)\wedge\partial_t A\\ &-\int_M p_t^{-1}\mathbf{h}_i\wedge\partial_t\mathrm{d}A\\ &-\int_{\partial M} H_\partial\wedge\partial_t A+\dot{W}_M\end{aligned}$$

and using on the other hand the results of Section II

$$\begin{aligned}&\partial_t\left(p_t^{-1}\Psi_M\right)(\mathrm{d}A, X)\\ &=\int_M\left(\partial_B p_t^{-1}\rho_M^\Psi\right)(\mathrm{d}A, X)\wedge\partial_t\mathrm{d}A\\ &+\int_M\left(\partial_t p_t^{-1}\rho_M^\Psi\right)(\mathrm{d}A, X)\end{aligned}$$

two expressions of the variation in time of the magnetic energy are obtained. Identification of the two right-hand sides gives an equation that must be always verified, i.e., for any (sub)manifold M and any variation of A. The implied conservation equations are obtained by identifying to zero the factors of $\partial_t A$, separately on M and ∂M. Using the properties $\int_{\partial M}\alpha = \int_M \mathrm{d}\alpha, \partial_t\mathrm{d} = \mathrm{d}\partial_t$ and

$$\mathrm{d}(\alpha\wedge\beta) = \mathrm{d}\alpha\wedge\beta + (-1)^{\deg(\alpha)}\alpha\wedge\mathrm{d}\beta \tag{5}$$

one obtains

$$\begin{aligned}&\mathrm{d}\left\{\left(\partial_{\mathbf{B}}\, p_t^{-1}\rho_M^\Psi\right)(\mathrm{d}A, X)+p_t^{-1}\mathbf{h}_i\right\} = J+\partial_t D \quad \text{on } M\\ &\left(\partial_{\mathbf{B}}\, p_t^{-1}\rho_M^\Psi\right)(\mathrm{d}A, X)+p_t^{-1}\mathbf{h}_i = H_\partial \quad \text{on } \partial M\end{aligned}$$

and the remaining of the equation writes

$$\dot{W}_M = \int_M\left(\partial_t p_t^{-1}\rho_M^\Psi\right)(\mathrm{d}A, X) \quad \text{on } M$$

Note that the undefined dependencies in X have not been expressed explicitly here, X will be assumed to be constant in the remaining of the paper.

Conservation relation at the other nodes of the diagram are obtained similarly. One finds finally

$$\mathrm{d}H = J+\partial_t D \tag{6}$$
$$E = -\partial_t A - \mathrm{d}U \tag{7}$$
$$p_t^{-1}\mathbf{e}_j + \alpha\mathcal{L}_{\mathbf{V}_c} J = -\partial_t A - \mathrm{d}U \tag{8}$$
$$0 = \mathrm{d}\left(J+\partial_t D\right) \tag{9}$$

with the *definitions*

$$H \equiv \left(\partial_B p_t^{-1}\rho_M^\Psi\right)(\mathrm{d}A, X) + p_t^{-1}\mathbf{h}_i \tag{10}$$
$$E \equiv \left(\partial_D p_t^{-1}\rho_E^\Psi\right)(D, X) + p_t^{-1}\mathbf{e}_i \tag{11}$$

the boundary condition $H = H_\partial$ on ∂M, and

$$\dot{W}_M = \int_M\left(\partial_t p_t^{-1}\rho_M^\Psi\right)(\mathrm{d}A, X) \tag{12}$$
$$\dot{W}_E = \int_M\left(\partial_t p_t^{-1}\rho_E^\Psi\right)(D, X) \tag{13}$$

which obviously, in the absence of any supplementary X dependency in ρ_M^Ψ, define the work delivered by electromagnetic forces.

Equations (6)–(9) can be mapped into E, thanks to the placement map p_t, so as to obtain the conservation equations, not in terms of differential forms, but in terms of the corresponding scalar and vector fields. Alternatively, the conservation equations can be derived directly from the diagram in E (Fig. 2, below) using vector field analysis. One obtains in Ω

$$\mathrm{curl}\mathbf{h} = \mathbf{j}+\mathcal{L}_\mathbf{v}\mathbf{d} \tag{14}$$
$$= -\mathcal{L}_\mathbf{v}\mathbf{a} - \mathrm{grad}\, u \tag{15}$$
$$\mathbf{e}_j + \alpha\mathcal{L}_{\mathbf{v}+\mathbf{v}_c}\mathbf{j} = -\mathcal{L}_\mathbf{v}\mathbf{a} - \mathrm{grad}\, u \tag{16}$$
$$0 = \mathrm{div}\left(\mathbf{j}+\mathcal{L}_\mathbf{v}\mathbf{d}\right) \tag{17}$$

with the *definitions*

$$\mathbf{h} \equiv \left(\partial_\mathbf{b}\rho_M^\Psi\right)(\mathrm{curl}\,\mathbf{a}) + \mathbf{h}_i \tag{18}$$
$$\mathbf{e} \equiv \left(\partial_\mathbf{d}\rho_E^\Psi\right)(\mathbf{d}) + \mathbf{e}_i \tag{19}$$

and the boundary condition $\mathbf{h} = \mathbf{h}_\partial$ on $\partial\Omega$. The work delivered by the electromagnetic forces writes now

$$\dot{W}_M = \int_\Omega\left(\mathcal{L}_\mathbf{v}\rho_M^\Psi\right)(\mathrm{curl}\,\mathbf{a}, x) \tag{20}$$
$$\dot{W}_E = \int_\Omega\left(\mathcal{L}_\mathbf{v}\rho_E^\Psi\right)(\mathbf{d}, x). \tag{21}$$

V. Discussion

A. Constitutive Laws

This formulation of electromagnetism gives back to the material manifold its fundamental place in the theory. Constitutive laws are defined by giving algebraic expressions for the energy density functionals $(\rho_M^\Psi, \rho_E^\Psi, \rho_K^\Psi)$ and to the dissipation functions $(\mathbf{h}_i, \mathbf{e}_i, \mathbf{e}_j)$. The conservation (6)–(9) or (14)–(17) do not contradict Maxwell's equations, but they are more complete and more detailed. All terms have a clear physical meaning in terms of energy or energy transfer. The different regimes (magnetostatics, electrodynamics,...) are easily characterized by cutting off one or several reservoirs in the diagram. Equation (9), resp. (17), is redundant with (6), resp. (14), as a consequence of the fact that the U-reservoir is empty.

B. Magnetic Field

Ampere's law (17) is found in this way to ensure conservation of magnetic energy. Equation (18) shows that, in the presence of dissipative phenomena, the magnetic field decomposes actually into a reversible part $\mathbf{h}_r \equiv \partial_\mathbf{b}\rho_M^\Psi$ that accounts for the magnetization phenomenon (alignment of microscopic magnetic moments), and an irreversible part $\mathbf{h}_i$ that accounts for the local dissipation process. The classical magnetic field $\mathbf{h}$ is thus a composite containers for phenomena of different natures. This clarifies the issue of hysteresis modeling by indicating that the state variable should be the induction (or a magnetization in Tesla) subjected to a force deriving from a potential (the magnetic energy) and a dissipative force.

C. Forces and Motion

By setting $\mathcal{L}_{\mathbf{v}}\mathbf{a} = 0$, the A-reservoir is isolated from the diagram. When the system deforms, the variation of the energy of that reservoir is therefore necessarily the mechanical power delivered by the magnetic forces. Similar considerations hold for the D-reservoir. The work delivered by electromagnetic forces can then be defined by

$$\dot{W}_M = \partial_t \Psi_M \,|_{\mathcal{L}_{\mathbf{v}}\mathbf{a}=0} \qquad \dot{W}_E = \partial_t \Psi_E \,|_{\mathcal{L}_{\mathbf{v}}\mathbf{d}=0} \qquad (22)$$

which is equivalent to the already mentioned definitions (20) and (21). By using (2)–(3) and factorizing $\nabla\mathbf{v}$, (22) leads straightforwardly to the definition of the Maxwell stress tensor of the material, which is the fundamental quantity representing the electromechanical coupling [2], [3]. Motion terms like $\mathbf{v}\times\mathbf{b}$ are explicitly present by virtue of the material derivative (2), and need not be introduced on basis of a relativistic argument. Relativity is still an issue but it applies only to the choice of the referential in E.

D. Electric Field

As mentioned above for the magnetic field $\mathbf{h}$, the electric field $\mathbf{e}$ (19) is also a composite container for phenomena of different nature. But the situation is for the latter still more confusing. Indeed, (15) and (16) give

$$\mathbf{e} = \left(\partial_{\mathbf{d}}\rho_E^{\Psi}\right)(\mathbf{d}) + \mathbf{e}_i(\partial_t \mathbf{d}) \qquad (23)$$

$$= -\mathcal{L}_{\mathbf{v}}\mathbf{a} - \operatorname{grad} u = \mathbf{e}_j(\mathbf{j}) + \alpha\mathcal{L}_{\mathbf{v}+\mathbf{v}_c}\mathbf{j} \qquad (24)$$

which represents not less than three equivalent expressions of the electric field.

Equation (16) in particular is the equilibrium equation for charge carriers, up to a factor q_c. The term $-\operatorname{grad} u$ is the applied electrostatic force and the term $\mathbf{e}_i(\mathbf{j}) = \sigma^{-1}\mathbf{j}$ is the viscous force opposed by the crystal lattice. When the charge carrier accelerates, a certain amount of energy has to be given to increase its kinetic energy and another amount of energy to increase the magnetic energy of the system, as the accelerated particle is associated with a larger current, which in turn generates a larger magnetic field. These two energy transfers are respectively represented by the forces (up to the factor q_c again) $\alpha\mathcal{L}_{\mathbf{v}+\mathbf{v}_c}\mathbf{j}$ and $\partial_t A$ that can be regarded as two inertial forces of different natures.

E. Superconductors

In practice, the J-reservoir can often be considered as empty as well, because of the very small value of α (negligible inertia of the charge carriers), and the corresponding term in (16) can be disregarded. However, in superconductors, for which σ is infinite ($\mathbf{e}_j = 0$) and $\operatorname{grad} u$ is zero, (16) reads

$$\alpha\mathcal{L}_{\mathbf{v}+\mathbf{v}_c}\mathbf{j} = -\mathcal{L}_{\mathbf{v}}\mathbf{a}. \qquad (25)$$

If the cloud of charge is not too much distorted, one has $\mathcal{L}_{\mathbf{v}} \equiv \partial_t$, so that London's equation for superconductors $\mathbf{a} = -\alpha\mathbf{j}$ is found back.

F. Electric Charges

The inertia of the charge carrier is also at the root of the definition of the static charges that are present at the surface of current carrying conductors [4]. Electric charge are first *defined* by

$$\rho^Q = \mathrm{d}D = \operatorname{div}\mathbf{d}. \qquad (26)$$

Note that they are not present in the diagram, neither in the conservation equations. Identifying the left hand sides of (15) and (17) and assuming $\mathbf{e}_i = 0$, one has

$$\left(\partial_{\mathbf{d}}\rho_E^{\Psi}\right)(\mathbf{d}) = \varepsilon_0^{-1}\mathbf{d} = \sigma^{-1}\mathbf{j} + \alpha\mathcal{L}_{\mathbf{v}+\mathbf{v}_c}\mathbf{j}. \qquad (27)$$

The divergence of the right-hand side is identically zero (div and $\mathcal{L}_{\mathbf{v}}$ commute) inside the conductor, but the term in α has a nonzero contribution on the surface of the conductor, whence the expression $\varepsilon_0\alpha\mathcal{L}_{\mathbf{v}+\mathbf{v}_c}\mathbf{j}\cdot\mathbf{n}$ of the surface charges.

G. Poynting's Relation

Poynting's relations is found by cutting off the J-reservoir from the diagram and noting that the two surface flows combine to form the flow of the Poynting vector (28)

$$\int_{\partial\Omega} \mathbf{h}_\partial \times \mathcal{L}_{\mathbf{v}}\mathbf{a} + u\left(\mathbf{j} + \mathcal{L}_{\mathbf{v}}\mathbf{d}\right)\} \cdot \mathbf{n} = \int_{\partial\Omega} \mathbf{e}\times\mathbf{h}. \qquad (28)$$

VI. Conclusion

The proposed energy-based approach considers the problem of electromagnetism in a continuous medium in all its generality. The only fundamental assumptions are the continuous medium approach and a sufficient differentiability of all quantity involved. Material aspects are integrated, not under the form of constitutive laws, but in terms of energy functionals and dissipative generalized forces. With the obtained energy diagram, several issues related with the interaction of EM fields with matter (e.g., hysteresis, forces, superconductors) find a clear presentation and a natural theoretical framework. The proposed diagram is of a great help to establish correct multiphysics models.

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