A comprehensive theory for electromagnetic force formulae in continuous media

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***Abstract*— An energy-based theory for electromagnetic forces in continuous media is presented, aiming at providing a complete toolbox for their numerical computation. In an Euclidean space, the electromechanical coupling is shown to be realised by a stress tensor, in terms of which all classical electromagnetic force formulae can be re-interpreted, unified, and sometimes generalised.**

I. Introduction

The existence of such a long controversy about the computation of electromagnetic forces and of so many uncertainties among the practitioners is certainly to ascribe to the fact that the issue cannot be completely clarified with the concepts of Vector analysis only. The analysis of this problem requires to consider a deforming body, and to apply energy conservation rules to it. Whereas the mathematical issue has been addressed by Alain Bossavit [1], [2], [3], [4], this paper provides to those who cannot invest much time in a detailed study of Differential geometry, the operative formalism that follows from the theoretical discussion, i.e. the set of formulae leading systematically from the very statement of the electromechanical problem up to a practical implementation of a solution method for it.

The first section states the transcription of the theoretical results in terms of classical vector and tensor fields and the co-moving time derivative of fields is introduced. This transcription reveals that the electromechanical coupling term, i.e. the volume density of mechanical power $\dot{\rho}^W_{em}$ developed by electromagnetic forces, can always be expressed in terms of a stress tensor σ_{em}. The purpose of the paper is to describe the procedure and the rules to determine in practical situations the algebraic expression of this tensor.

The second section shows that all classical forces formulae for rigid bodies or local forces, e.g. [5], [6], [7], can be derived quite straightforwardly from the general principles stated in the previous section, by choosing in each case a particular velocity field. This is not only a backwards result but also a solid departure point to tackle with more complex materials. The case of nonlinear anisotropic material, e.g. iron or steel, is treated in detail.

Finally, the third section, relying on the energy-based formulation of the Maxwell system of equation presented in [8], deals with the question of electromagnetic forces in the presence of Joule losses, hysteresis and magnetostriction.

II. Co-moving time derivative of fields

Each kind of field in a 3D space is inherently attached to particular geometrical entities of the underlying manifold (domain), either points, curves, surfaces or volumes. They are accordingly named 0–, 1–, 2– or 3 form. When the manifold flows or deforms with time, supplementary terms of geometrical origin must be added to the partial time derivative. Differential geometry tells that the time derivative $\mathcal{L}_{\mathbf{v}}$ associated with the velocity field $\mathbf{v}$ describing the flow/deformation is given, in terms of classical euclidian tensor analysis, by

$$\mathcal{L}_{\mathbf{v}} f = \dot{f} \quad (1)$$

$$\mathcal{L}_{\mathbf{v}} \mathbf{h} = \dot{\mathbf{h}} + (\nabla \mathbf{v}) \cdot \mathbf{h} \quad (2)$$

$$\mathcal{L}_{\mathbf{v}} \mathbf{b} = \dot{\mathbf{b}} - \mathbf{b} \cdot (\nabla \mathbf{v}) + \mathbf{b}\ \mathrm{tr}(\nabla \mathbf{v}) \quad (3)$$

$$\mathcal{L}_{\mathbf{v}} \rho = \dot{\rho} + \mathrm{tr}(\nabla \mathbf{v})\ \rho \quad (4)$$

respectively for 0–, 1–, 2– and 3–forms, where $\dot{z} = \partial_t z + \mathbf{v} \cdot \nabla z$ denotes the *total derivative* of $z(t, x^k)$, applied component by component if z is a vector field. Whereas (1) and (4) are classical in fluid dynamics, where they are called *material derivatives*, (2) and (3) are less often encountered and apply precisely to electromagnetic fields. As the latter do not need material support, $\mathcal{L}_{\mathbf{v}}$ is called *co-moving time derivative* [9].

Considering now a material for which the electromagnetic energy density is a function of induction and strain, $\rho^\Psi(\mathbf{b}, \varepsilon)$ (other dependencies do not affect the calculated forces), it will be shown in the full paper that one has

$$\rho^{\dot{W}}_{em} = -\sigma_{em} : \nabla \mathbf{v} \quad (5)$$

with the Maxwell stress tensor

$$\sigma_{em} = \mathbf{b}\ \tilde{\mathbf{h}} + \partial_\varepsilon \rho^\Psi - (\tilde{\mathbf{h}} \cdot \mathbf{b} - \rho^\Psi)\mathbb{I} \quad (6)$$

defined as the factor of $\nabla \mathbf{v}$ and $\tilde{\mathbf{h}} \equiv \partial_{\mathbf{b}} \rho^{\dot{W}}_{em}$. Note the use of the dyadic (undotted) vector product $(\mathbf{v}\ \mathbf{w})_{ij} = v^i w^j$, the tensor product $a : b = a_{ij} b_{ij}$ and the identity matrix $\mathbb{I}$.

Equations (5) and (6) are the fundamental equations that will be discussed in the full paper. It will be demonstrated that the definition of forces for any kind of material reduces to the correct (i.e. corresponding with reality) definition of the energy functional ρ^Ψ.

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