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The structure of EM energy flows in continuous media

François Henrotte and Kay Hameyer
RWTH-Aachen University – Institute of Electrical Machines
Schinkelstraße 4, D-52056 Aachen, Germany
E-mail: Francois.Henrotte@IEM.RWTH-Aachen.de

***Abstract*—A formulation of electromagnetic problems in continuous media is proposed, which relies on a clear identification of the different existing electromagnetic energy reservoirs and the flows between them. A rich structure is revealed, which constitutes a natural framework to establish the differential and finite element equations describing a given problem. This formulation, which unlike Maxwell's equations integrates also material aspects, clarifies several issues related to dissipative and coupled phenomena in magnetic materials.**

I. Introduction

Maxwell's equations are generally presented as the fundamental set of equations ruling all electromagnetic (EM) phenomena. However they address only a part of the question, as they need to be complemented by constitutive laws. Moreover, such a splitting of the problem into a material part and a non-material part is done at the cost of a series of conditions that are scarcely stated explicitly and have however important implications.

II. Theoretical setup

The theoretical framework we need relies upon two manifolds with distinct functions: the material manifold M of which each point is associated with a material particle of the continuous medium (e.g. an atom), and the Euclidean space E which represents the space where the motion takes place and which is a manifold where a metric has been defined.

In order to describe a possible movement or deformation of the system, the placement map $p_t : X \in M \mapsto x = p_t X \in E$ is defined. It is a map that associates its position in E to each material particle $X \in M$ at all instants of time $t \in [t_A, t_B]$. The codomain of the placement map, $\Omega = p_t M$, is the deformed state. On the other hand, the codomain of the map $t \in [t_A, t_B] \mapsto x = p_t X \in E$ is the trajectory of a particular material particle X (Fig. 1). The velocity field, $\mathbf{v} = \partial_t x$ (vectors in E are denoted with a bold letter), is the field of tangent vectors to all trajectories of the flow at a given instant of time.

The placement p_t is assumed to be regular and invertible at all t. It induces a 1-1 mapping, also noted p_t, of all vector and tensor fields defined on M to the corresponding fields defined on E. If quantities defined on M are denoted with an uppercase symbol, and quantities defined on E are denoted with a lowercase, one has $p_t Z = z$.

Electromagnetic fields are by definition differential forms defined on M. Unconventionnally, this approach is not in terms of the classical $\mathbf{h}, \mathbf{b}, \mathbf{e}$ and $\mathbf{d}$ fields, but in terms of the electric scalar potential U (0–form), the magnetic vector potential A (1–form), the electric displacement D (2–form) and the current density J (2–form). Finally, the magnetic field H_∂ on the boundary ∂M of M is also required as a variable, so as to complete the boundary conditions.

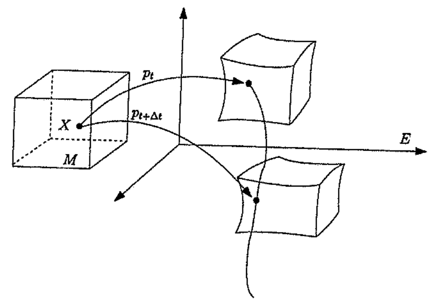

Fig. 1. Placement map at two instants of time and the trajectory of X in E.

The metric on E allows to attribute an intensity to the fields defined in M, thanks to p_t. For instance, the magnetic flux density writes $\mathrm{d}A$ in M, since the exterior derivative d is the differential geometry equivalent of the curl operator, i.e. $p_t \, \mathrm{d}A = \mathrm{curl}\, \mathbf{a}$. It associates a flux φ (in Weber) to any infinitesimal material surface Σ in M. But one needs the measure of $p_t\Sigma$ in E, and hence the metric on E, to determine the intensity of the field $\varphi/\mathrm{measure}(p_t\Sigma)$. The magnetic energy density is thus a function of $\mathrm{curl}\, \mathbf{a}$ (not of $\mathrm{d}A$), and of possible other arguments like temperature, strain, etc. If the magnetic energy is noted Ψ_M and its corresponding density ρ_M^Ψ (the density of any quantity X is denoted ρ^X), one has

$$\Psi_M(\mathrm{curl}\, \mathbf{a}, \ldots) \equiv \int_\Omega \rho_M^\Psi(\mathrm{curl}\, \mathbf{a}, \ldots) = (p_t^{-1}\Psi_M)(\,\mathrm{d}A, \ldots).$$

Again, p_t gives the expression of the energy in M corresponding to the one given in E. Details will be given in the full paper. Identical considerations apply to the electric energy $\Psi_E(\mathbf{d}, \ldots)$.

III. EM energy flow diagram

The topology of the EM energy flow diagram, as depicted on Fig. 2, is a square with at each corner a reservoir associated with a particular field, resp. A,D,J,U from the upper left to the lower right corner. The $A-$reservoir and the $D-$reservoir contain resp. magnetic and electric energy. The $J-$reservoir contains the kinetic energy of the charge carriers. Let M_c denote the mass of one charge carrier and Q_c its charge. If ρ_c is the density of charge carriers, which we shall assume constant, and $\mathbf{v}_c$ their velocity field in E, the current density is $\mathbf{j} = Q_c \rho_c \mathbf{v}_c$. The kinetic energy density writes then $\rho_K^\Psi(\mathbf{j}) =$

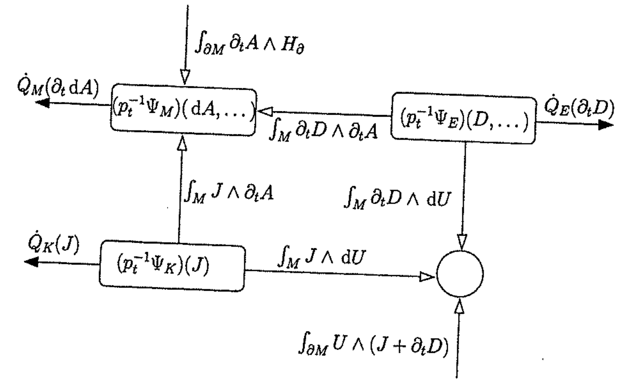

Fig. 2. EM energy flow diagram in the material manifold M.

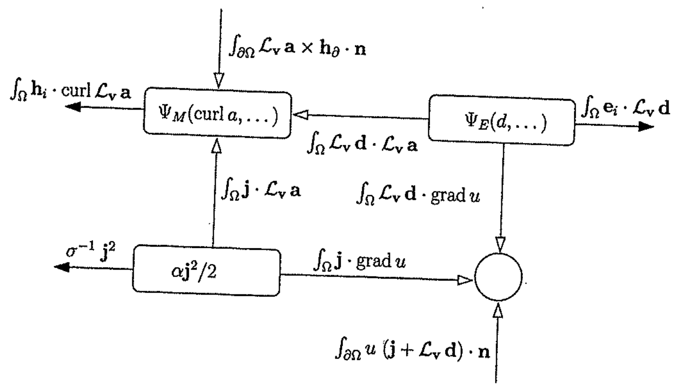

Fig. 3. EM energy flow diagram in the euclidean space E.

$\rho_c M_c \mathbf{v}_c^2/2 = \alpha\mathbf{j}^2/2$ with $\alpha = M_c/(\rho_c Q_c^2)$. Finally, the $U-$reservoir is always empty.

The reservoirs exchange energy with each other by means of internal energy flows, which are completely determined by the knowledge of A, D, J and U. They can also exchange energy with the exterior, either through the surface of the system (surface terms at nodes A and U) or by bulk dissipation (black-headed arrows).

IV. Conservation Equations

As the fields A, D, J and U are independent of each other, they can be varied freely in order to obtain, by a simple variational argument, the conservation equations implied by the structure of the EM energy flow diagram. Before doing so, however, the whole structure is mapped into E, thanks to the placement map p_t, so as to obtain the equations, not in tems of differential forms, but in terms of the corresponding scalar and vector fields. In the mapping process, the commutation property $p_t\partial_t = \mathcal{L}_\mathbf{v} p_t$ is used, where $\mathcal{L}_\mathbf{v}$ is the material derivative. The dissipation functions have also been given more familar expressions, which are not restrictive (See Fig. 3): $\mathbf{h}_i$ is a function of $\mathcal{L}_\mathbf{v}\,\mathrm{curl}\,\mathbf{a}$, $\mathbf{e}_i$ is a function of $\mathcal{L}_\mathbf{v}\,\mathbf{d}$.

In Ω, the conservation equations obtained by varying the four independent fields $\mathbf{a}$, $\mathbf{d}$, $\mathbf{j}$ and u write respectively

$$\mathrm{curl}\,\{\partial_\mathbf{b}\rho_M^\Psi(\mathrm{curl}\,\mathbf{a}) + \mathbf{h}_i\} = \mathbf{j} + \mathcal{L}_\mathbf{v}\,\mathbf{d} \quad (1)$$

$$\partial_\mathbf{d}\rho_E^\Psi(\mathbf{d}) + \mathbf{e}_i = -\mathcal{L}_\mathbf{v}\,\mathbf{a} - \mathrm{d}u \quad (2)$$

$$\sigma^{-1}\mathbf{j} + \alpha\mathcal{L}_{\mathbf{v}_c}\,j = -\mathcal{L}_\mathbf{v}\,\mathbf{a} - \mathrm{d}u \quad (3)$$

$$0 = \mathrm{div}\,(\mathbf{j} + \mathcal{L}_\mathbf{v}\,\mathbf{d}) \quad (4)$$

complemented with the boundary condition $\mathbf{h}_\partial = \mathbf{h}_r + \mathbf{h}_i$ on $\partial\Omega$. Note that in the absence of motion, $\mathbf{v} \equiv 0$ and $\mathcal{L}_\mathbf{v} \equiv \partial_t$.

V. Discussion

This approach gives back to the material manifold its fundamental and unique place in the theory. Relativity is still an issue but it applies only to the choice of the referential in E. Constitutive laws are defined by giving analytic expressions for the energy functions Ψ_X in and the dissipation functions $\dot{Q}_X$. Of course, the conservation equations (1–4) do not contradict Maxwell's equations, but they are more detailed. All terms have a clear physical meaning in terms of energy or energy transfer. The different regimes (Magnetostatics, Electrodynamics,...) are readily characterised by cutting off one or several reservoirs in the diagram.

Motion terms like $\mathbf{v} \times \mathbf{b}$ are explicitly present by virtue of the material derivative, and need not be introduced on basis of a foreign relativistic argument. The definition of EM forces [1], [2] is now an immediate and obvious consequence of the structure of the diagram. The expression of the material derivatives of the different kind of fields will be given in the full paper.

Comparison of Ampere's law with (1) shows that, in the presence of dissipative phenomena, the magnetic field $\mathbf{h}$ decomposes actually into a reversible part $\partial_\mathbf{b}\rho_M^\Psi(\mathrm{curl}\,\mathbf{a})$ which accounts for the magnetisation phenomenon (alignment of microscopic magnetic moments), and an irreversible part $\mathbf{h}_i$ which accounts for the local dissipation process. The magnetic field $\mathbf{h}$, as well as the electric field $\mathbf{e}$, is thus a composite container for phenomena of different natures. This quite clarifies the issue of hysteresis modelling [3].

Equation (4) is redundant with (1) as a consequence of the fact that the $u-$reservoir is empty. In practice, the $\mathbf{j}-$reservoir can also be considered as empty, due to the very small value of α and the corresponding term in (3) can be disregarded. However, in superconductors, for which σ is infinite and $\mathrm{grad}\,u$ is zero, (3) reads $\mathbf{a} = -\alpha\mathbf{j}$, which is London's model. That term is also at the root of the definition of the static charges on the boundaries of current carrying conductors [4], which can be evaluated by $\varepsilon_0\alpha\mathcal{L}_{\mathbf{v}_c}\,\mathbf{j}\cdot\mathbf{n}$.

VI. Conclusions

The proposed energy-based approach considers the problem of electromagnetism in a continuous medium in all its generality. Material aspects are integrated, not under the form of constitutive laws, but in terms of energy and dissipation functionals. With the obtained energy diagram, several issues related with the interaction of EM fields with matter (hysteresis, forces, superconductors, ...) find a clear explanation and a natural theoretical framework.

References

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