# Electromagnetic Force Densities in a Continuous Medium

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**Abstract.** The paper introduces a systematic procedure to derive the expression of the Maxwell stress tensor associated with a given expression of the electromagnetic energy density.

## 1 Introduction

There exist numerous formulae for the computation of electromagnetic (EM) forces. They can be sorted into two distinct families, depending on whether they are based on the definition of a *Maxwell stress tensor* or on the application of the *virtual work principle.* A look into the literature shows that expressions of *Maxwell stress tensors* (See e.g. [1]) are generally obtained by algebraic and differential operations, starting from the Maxwell equations and assuming an *a priori* knowledge of the expression of the force density in the medium under consideration. The *virtual work principle*, on the other hand, relies more clearly in theory on the required energy concepts [2–4], but the formulae proposed in [5–7] for its implementation are all obtained by a roundabout way involving the jacobian matrix of a mapping at the finite element level. In both cases, the underlying thermodynamic concepts are buried into an overwhelming algebra.

## 2 The Euclidean case

The first step towards a thermodynamic analysis of an electromechanical system is to define the magnetic and mechanical state variables in such a way that they are independent of each other. Whereas it sounds obvious to anybody that one can freely modify the magnetic field [1] in a system without deforming it, by increasing the imposed currents for instance, it is much less clear to imagine how the system can be deformed without modifying the magnetic field. One feels indeed that any deformation of the system will affect the magnetic field.

Let first $M$ be the *material manifold*, i.e. a continuous set of points each representing a material particle of a given electromechanical system. Let $C_1(M)$ be the set of all regular curves in $M$ and $C_2(M)$ be the set of all

[1] Considered in this section as a vector field, *not* as a differential form.

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regular surfaces. Let $E$ be the Euclidean space $\mathbb{R}^3$. Following [8], the magnetic state variable of the electromechanical system is defined by the *magnetic flux map*

$$\phi : C_2(M) \mapsto \mathbb{R} \tag{1}$$

that associates a real number, the *magnetic flux*, to any surface in $M$. Similarly, the kinematics of the system is defined by the *placement map*

$$p : M \mapsto \Omega \subset E, \tag{2}$$

which associates to any point of $M$ its position in $E$.

Although the magnetic flux map (1) determines completely the fluxes in the system, it does not give the local value of the *induction field.* The latter is a secondary quantity that requires the definition of an *interpolation operator* noted $\boldsymbol{b}(\phi, p)$, as it may depend on both $p$ and $\phi$. This is the reason why the interpolated induction field is not suitable as a primary variable in a thermodynamic representation. The properties of this vector-valued interpolation operator are not trivial. It may involve, for instance, the selection of a set of particular facets for the representation of the field, and an accuracy and convergence analysis. The interpolation with Whitney facet elements in a mesh is an example of such an interpolation tool. Another example is given below. A similar interpolation operator, denoted by $\boldsymbol{x}(p)$, is associated with the placement map.

Since the maps $\phi$ and $p$ are independent of each other, they are suitable variables for the definition of the *energy functional* of the system $\Omega$. One has

$$\Psi(\phi, p) = \int_{\Omega(p)} \rho^{\Psi}(\boldsymbol{b}(\phi, p), \boldsymbol{x}(p), p) \tag{3}$$

where the *energy density* $\rho^{\Psi}$ depends on the *interpolated* vector fields $\boldsymbol{b}$ and $\boldsymbol{x}$.

If the problem is more easily posed in terms of the magnetic field $\boldsymbol{h}$, than in terms of $\boldsymbol{b}$, the available thermodynamic state function is the *coenergy functional*

$$\Phi(I, p) = \int_{\Omega(p)} \rho^{\Phi}(\boldsymbol{h}(I, p), \boldsymbol{x}(p), p) \tag{4}$$

with

$$I : C_1(M) \mapsto \mathbb{R} \tag{5}$$

the *magnetomotive force map*, which associates a real number, the *magnetomotive force*, to any curve in $M$, and $\boldsymbol{h}(I, p)$ the interpolation operator for the magnetic field.

The definition of forces follows now from the variation of those energy functionals, $\delta\Psi(\phi, p)|_{\delta\phi=0}$ or $\delta\Phi(I, p)|_{\delta I=0}$, and the factorization under the