Calculation of the 3D Non-linear Eddy Current Field in Moving Conductors and its Application to Braking Systems

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***Abstract*— This paper presents a complete FE formulation for the calculation of 3D non-linear eddy current fields with ferromagnetic moving conductors. The formulation is realized by using the $\vec{A}$, V-$\vec{A}$ formulation in combination with the Coulomb gauge. To consider non-rectangular shapes of geometries tetrahedral elements were employed. The computation procedure is applied to an eddy current braking system of a high velocity train and the resulting braking forces are compared to measurements.**

I. Introduction

The calculation of the 3D eddy current field of moving ferromagnetic conductors is necessary for the design of eddy current braking systems (see Fig. 1) used in high velocity or magnetic levitation trains. For that reason a software with a FD approach had been used [1]. But the rising demand of a correct modeling of real shapes of brakes and rails showed the limitations of that method. This led to the following realization of these computations by the FE method with tetrahedral elements.

Most papers concerning this subject use the $\vec{A}$, V formulation for the moving parts, e.g. [2],[3]. Little attention has been given to non-linear 3D calculations and their agreement to measurements .

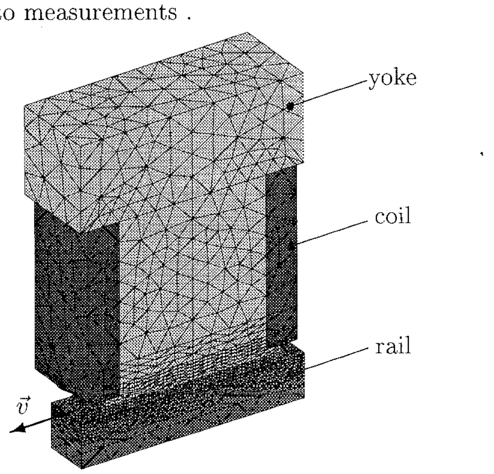

Fig. 1. Longitudinal section view of a pole of an eddy current brake

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II. Calculation Method

A. Problem Definition

Fig. 2 shows a simple configuration of three regions causing an electromagnetic field problem with eddy currents due to motion. One region Ω_1 is moving with a

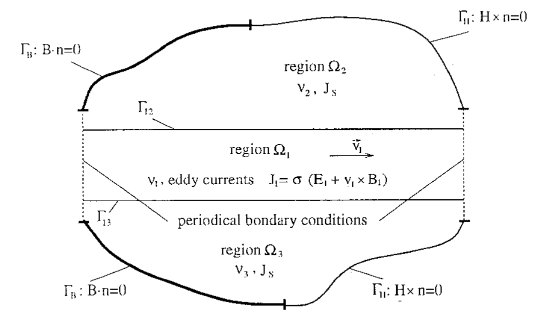

Fig. 2. Magnetostatic field problem with a moving conductor

relative velocity to the surrounding regions Ω_2 and Ω_3. Assuming the geometry to be invariant in the direction of motion the Maxwell equations for this field problem are:

$$\left.\begin{aligned}
\operatorname{curl}\vec{E} &= \vec{0} && (1)\\
\operatorname{div}\vec{J} &= 0 && (2)\\
\vec{J} &= \sigma\left(\vec{E}+\vec{v}\times\vec{B}\right) && (3)\\
\operatorname{curl}\vec{H} &= \vec{J} && (4)\\
\operatorname{div}\vec{B} &= 0 && (5)\\
\vec{H} &= \nu(B)\,\vec{B} && (6)
\end{aligned}\right\}\text{ in }\Omega_1$$

In the following only the regions Ω_1 and Ω_2 will be considered, because the mathematical formulations for region Ω_2 and Ω_3 are the same. In region Ω_2 only the magnetic part of the electromagnetic field has to be solved, because in a static field the electric field does not depend on the magnetic field:

$$\left.\begin{aligned}
\operatorname{curl}\vec{H} &= \vec{J}_s && (7)\\
\operatorname{div}\vec{B} &= 0 && (8)\\
\vec{H} &= \nu(B)\,\vec{B} && (9)
\end{aligned}\right\}\text{ in }\Omega_2$$

The interface boundary conditions on Γ_{12} are the continuity of the normal component of the flux density $\vec{B}$ and the current density $\vec{J}$ and of the tangential component of the magnetic field intensity $\vec{H}$:

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$$\left.\begin{array}{rcl} \vec{n} \times \left(\vec{H}_1 - \vec{H}_2\right) & = & \vec{0} \\ \vec{n} \cdot \left(\vec{B}_1 - \vec{B}_2\right) & = & 0 \\ \vec{n} \cdot \vec{J}_1 & = & 0 \end{array}\right\} \text{on } \Gamma_{12} \qquad \begin{array}{r}(10)\\(11)\\(12)\end{array}$$

The boundary of Ω_2 is devided into two parts. On Γ_B the boundary condition is described by a vanishing normal component of the flux density, on Γ_H by a vanishing tangential component of the magnetic field intensity:

$$\vec{n} \cdot \vec{B} = 0 \quad \text{on } \Gamma_B \qquad (13)$$

$$\vec{n} \times \vec{H} = \vec{0} \quad \text{on } \Gamma_H \qquad (14)$$

The periodical boundary conditions as well on the boundary of the moving parts as on the boundary of region Ω_2 are not considered in the following mathematical formulation but are easily to enforce in the numerical implementation.

B. Formulation

In order to satisfy equation (5) and (8) the magnetic vector potential $\vec{A}$ is introduced in Ω_1 and Ω_2. Additionally the electrical scalar potential V is used in Ω_1 to satisfy (1):

$$\vec{B} = \operatorname{curl} \vec{A} \quad \text{in } \Omega_1 \text{ and } \Omega_2 \qquad (15)$$

$$\vec{E} = -\operatorname{grad} V \quad \text{in } \Omega_1 \qquad (16)$$

This leads to an $\vec{A}$,V formulation in Ω_1 and an $\vec{A}$ formulation in Ω_2:

$$\operatorname{curl} \nu \operatorname{curl} \vec{A} - \left(\sigma \vec{v} \times \operatorname{curl} \vec{A} - \sigma \operatorname{grad} V\right) = \vec{0} \quad \text{in } \Omega_1 \qquad (17)$$

$$\operatorname{curl} \nu \operatorname{curl} \vec{A} = \vec{J}_s \quad \text{in } \Omega_2 \qquad (18)$$

Using these formulations the boundary conditions (10)-(14) on Γ_{12}, Γ_B and Γ_H can be written as:

$$\left.\begin{array}{rcl} \vec{n} \times \left(\nu_1 \operatorname{curl} \vec{A}_1 - \nu_2 \operatorname{curl} \vec{A}_2\right) & = & \vec{0} \\ \vec{n} \cdot \left(\operatorname{curl} \vec{A}_1 - \operatorname{curl} \vec{A}_2\right) & = & 0 \end{array}\right\} \text{on } \Gamma_{12} \qquad \begin{array}{r}(19)\\(20)\end{array}$$

$$\vec{n} \cdot \operatorname{curl} \vec{A} = 0 \quad \text{on } \Gamma_B \qquad (21)$$

$$\vec{n} \times \nu \operatorname{curl} \vec{A} = \vec{0} \quad \text{on } \Gamma_H \qquad (22)$$

Note that taking the divergence of (17) yields

$$\operatorname{div} \left(\sigma \vec{v} \times \operatorname{curl} \vec{A} - \sigma \operatorname{grad} V\right) = 0 \quad \text{in } \Omega_1 \qquad (23)$$

and consequently implies (2). Substracting the normal component of (17) and (18) on Γ_{12} leads to

$$\vec{n} \cdot \left(\sigma \vec{v} \times \operatorname{curl} \vec{A}_1 - \sigma \operatorname{grad} \vec{V}_1\right) = 0 \quad \text{on } \Gamma_{12} \qquad (24)$$

in view of the fact that no source current density of Ω_2 is floating into Ω_1 and that the other terms vanish by boundary condition (19) with the transformation

$$\begin{aligned} & \vec{n}_{12} \cdot (\operatorname{curl} \nu_1 \operatorname{curl} \vec{A}_1 - \operatorname{curl} \nu_2 \operatorname{curl} \vec{A}_2) \\ = \; & \operatorname{div} \left[\vec{n}_{12} \times (\nu_2 \operatorname{curl} \vec{A}_2 - \nu_1 \operatorname{curl} \vec{A}_1)\right] . \end{aligned} \qquad (25)$$

C. Uniqueness of the potentials in the $\vec{A}$,V -$\vec{A}$ formulation

Eq. (17) - (22) do not result in an unique solution for the potentials satisfying (1)-(14). With (15) only the curl of $\vec{A}$ is given. To ensure the uniqueness of the potentials $\vec{A}$ and V, the divergence of $\vec{A}$ has to be defined as well as the normal or tangential component of $\vec{A}$ on the boundary. A well posed set of conditions is given in [4] with

$$\operatorname{div} \vec{A} = 0 \quad \text{in } \Omega_1 \text{ and } \Omega_2 \qquad (26)$$

$$\vec{A}_1 = \vec{A}_2 \quad \text{on } \Gamma_{12} \qquad (27)$$

$$\vec{n} \times \vec{A} = \vec{0} \quad \text{on } \Gamma_B \qquad (28)$$

$$\vec{n} \cdot \vec{A} = 0 \quad \text{on } \Gamma_H . \qquad (29)$$

Now, together with these equation, (17) - (22) represent an unique solution for (1) - (14). Eq. (26) is known as Coulomb gauge, (27) ensures the continuity of $\vec{A}$ on Γ_{12} and automatically satisfies boundary condition (20), (28) defines the tangential component of $\vec{A}$ on Γ_B and satisfies condition (21) and finally (29) defines the normal component of $\vec{A}$ without affecting (22).

This gauging is realized by appending (17) and (18) by the term $(-\operatorname{grad} \nu \operatorname{div} \vec{A})$ which results in assumption of a constant reluctivity in the vector Laplacian operator of $\vec{A}$ replacing the $(\operatorname{curl} \nu \operatorname{curl} \vec{A})$ term. Further equation (23) now has to be added, because it does not follow from (17) appended by $(-\operatorname{grad} \nu \operatorname{div} \vec{A})$:

$$\left.\begin{array}{rcl} \operatorname{curl} \nu \operatorname{curl} \vec{A} - \operatorname{grad} \nu \operatorname{div} \vec{A} - & & \\ \left(\sigma \vec{v} \times \operatorname{curl} \vec{A} - \sigma \operatorname{grad} V\right) & = & \vec{0} \\ \operatorname{div} \left(\sigma \vec{v} \times \operatorname{curl} \vec{A} - \sigma \operatorname{grad} V\right) & = & 0 \end{array}\right\} \text{in } \Omega_1 \qquad \begin{array}{r}(30)\\(31)\end{array}$$

$$\operatorname{curl} \nu \operatorname{curl} \vec{A} - \operatorname{grad} \nu \operatorname{div} \vec{A} = \vec{J}_s \quad \text{in } \Omega_2 \qquad (32)$$

But herewith it must be ensured that the term including $(\nu \operatorname{div} \vec{A})$ in (30) and (32) in fact vanishes. Taking the divergence of these equations and noting (31) as well as the fact that $\vec{J}_s$ has to be divergence free leads in both regions to the Laplacian equation for $(\nu \operatorname{div} \vec{A})$:

$$\Delta \nu \operatorname{div} \vec{A} = 0 \quad \text{in } \Omega_1 \text{ and } \Omega_2 \qquad (33)$$

Taking the normal component of (32) on Γ_H yields the homogenous Neumann boundary condition on $(\nu \operatorname{div} \vec{A})$

$$\frac{\partial}{\partial n} \nu \operatorname{div} \vec{A} = 0 \quad \text{on } \Gamma_H \qquad (34)$$

by noting that $\vec{n} \cdot (\operatorname{curl} \nu \operatorname{curl} \vec{A})$ vanishes on Γ_H because of (22) with the transformation

$$\vec{n} \cdot (\operatorname{curl} \nu \operatorname{curl} \vec{A}) = -\operatorname{div}(\vec{n} \times \nu \operatorname{curl} \vec{A}) . \qquad (35)$$

Enforcing homogenous Dirichlet boundary conditions along Γ_B

$$\nu \operatorname{div} \vec{A} = 0 \quad \text{on } \Gamma_B \qquad (36)$$

and the continuity of $(\nu \operatorname{div} \vec{A})$ on Γ_{12}

$$\nu_1 \operatorname{div} \vec{A}_1 - \nu_2 \operatorname{div} \vec{A}_2 = 0 \quad \text{on } \Gamma_{12} \tag{37}$$

additionally to (33) and (34) yields the vanishing of $(\nu \operatorname{div} \vec{A})$ in Ω_1 and Ω_2.

Finally an $\vec{A}$,V -$\vec{A}$ formulation ensuring the uniqueness of $\vec{A}$ and V for the magnetostatic field problem with moving conductors is given as follows:

$$\left.\begin{aligned} \operatorname{curl} \nu \operatorname{curl} \vec{A} - \operatorname{grad} \nu \operatorname{div} \vec{A} - & \\ \left(\sigma \vec{v} \times \operatorname{curl} \vec{A} - \sigma \operatorname{grad} V\right) &= \vec{0} \quad (38) \\ \operatorname{div}\left(\sigma \vec{v} \times \operatorname{curl} \vec{A} - \sigma \operatorname{grad} V\right) &= 0 \quad (39) \end{aligned}\right\} \text{in } \Omega_1$$

$$\operatorname{curl} \nu \operatorname{curl} \vec{A} - \operatorname{grad} \nu \operatorname{div} \vec{A} - \vec{J}_s = \vec{0} \quad \text{in } \Omega_2 \tag{40}$$

$$\left.\begin{aligned} \vec{A}_1 &= \vec{A}_2 \quad (41) \\ \vec{n}_{12} \times \left(\nu_1 \operatorname{curl} \vec{A}_1 - \nu_2 \operatorname{curl} \vec{A}_2\right) &= \vec{0} \quad (42) \\ \nu_1 \operatorname{div} \vec{A}_1 - \nu_2 \operatorname{div} \vec{A}_2 &= 0 \quad (43) \\ \vec{n}_{12}\left(\sigma \vec{v} \times \operatorname{curl} \vec{A}_1 - \sigma \operatorname{grad} V_1\right) &= 0 \quad (44) \end{aligned}\right\} \text{on } \Gamma_{12}$$

$$\left.\begin{aligned} \vec{n} \times \vec{A} &= \vec{0} \quad (45) \\ \nu \operatorname{div} \vec{A} &= 0 \quad (46) \end{aligned}\right\} \text{on } \Gamma_B$$

$$\left.\begin{aligned} \vec{n} \times \nu \operatorname{curl} \vec{A} &= \vec{0} \quad (47) \\ \vec{n} \cdot \vec{A} &= 0 \quad (48) \end{aligned}\right\} \text{on } \Gamma_H$$

D. Numerical Implementention

The numerical approach of the analytic potentials $\vec{A}$ and V is based on a FE mesh with first order tetrahedral elements. The potential approximation in one element can be written as the product of the potential solutions on the nodes with the interpolation functions.

$$\vec{A} = \sum_{j=1}^{4} \left(A_{x_j} N_j \vec{e}_x + A_{y_j} N_j \vec{e}_y + A_{z_j} N_j \vec{e}_z\right) = \sum_{i=1}^{12} A_i \vec{N}_i \tag{49}$$

$$V = \sum_{i=1}^{4} V_i N_i \tag{50}$$

Applying the Galerkin weighted residual method with the shape functions as weighted functions to equation (38) - (40) leads to

$$\int_{\Omega_1} \vec{N}_i \cdot \left(\operatorname{curl} \nu \operatorname{curl} \vec{A} - \operatorname{grad} \nu \operatorname{div} \vec{A} - \left(\sigma \vec{v} \times \operatorname{curl} \vec{A} - \sigma \operatorname{grad} V\right)\right) d\Omega = 0 \tag{51}$$

$$-\int_{\Omega_1} N_i \cdot \operatorname{div}\left(\sigma \vec{v} \times \operatorname{curl} \vec{A} - \sigma \operatorname{grad} V\right) d\Omega = 0 \tag{52}$$

$$\int_{\Omega_2} \vec{N}_i \cdot \left(\operatorname{curl} \nu \operatorname{curl} \vec{A} - \operatorname{grad} \nu \operatorname{div} \vec{A} - \vec{J}_s\right) d\Omega = 0 \tag{53}$$

Using vector identities and integral transformations the volume integrals with second derivates of the potential functions can be transformed into volume and surface integrals with first order derivates, so that the equations (51)- (53) look as follows:

$$\begin{aligned} &\int_{\Omega_1} \Big(\nu \operatorname{curl} \vec{N}_i \cdot \operatorname{curl} \vec{A} + \nu \operatorname{div} \vec{N}_i \cdot \operatorname{div} \vec{A} \\ &\quad - \vec{N}_i \cdot \sigma \vec{v} \times \operatorname{curl} \vec{A} + \vec{N}_i \cdot \sigma \operatorname{grad} V\Big) d\Omega \\ &\quad - \int_{\Gamma_{12}} \vec{N}_i \cdot \left(\nu_1 \operatorname{curl} \vec{A}_1 \times \vec{n}_{12}\right) d\Gamma \\ &\quad - \int_{\Gamma_{12}} \vec{N}_i \cdot \vec{n}_{12}\, \nu_1 \operatorname{div} \vec{A}_1\, d\Gamma = 0 \end{aligned} \tag{54}$$

$$\begin{aligned} &\int_{\Omega_1} \sigma \operatorname{grad} N_i \cdot \vec{v} \times \operatorname{curl} \vec{A} - \sigma \operatorname{grad} N_i \cdot \operatorname{grad} V\, d\Omega \\ &\quad - \int_{\Gamma_{12}} N_i \left(\sigma \vec{v} \times \operatorname{curl} \vec{A}_1 - \sigma \operatorname{grad} V_1\right) \cdot \vec{n}_{12}\, d\Gamma = 0 \end{aligned} \tag{55}$$

$$\begin{aligned} &\int_{\Omega_2} \left(\nu \operatorname{curl} \vec{N}_i \cdot \operatorname{curl} \vec{A} + \nu \operatorname{div} \vec{N}_i \cdot \operatorname{div} \vec{A} - \vec{J}_s\right) d\Omega \\ &\quad - \int_{\Gamma_{12}} \vec{N}_i \cdot \left(\nu_2 \operatorname{curl} \vec{A}_2 \times \vec{n}_{21}\right) d\Gamma \\ &\quad - \int_{\Gamma_{12}} \vec{N}_i \cdot \vec{n}_{21}\, \nu_2 \operatorname{div} \vec{A}_2\, d\Gamma \\ &\quad - \int_{\Gamma_B} \vec{N}_i \cdot \left(\nu \operatorname{curl} \vec{A} \times \vec{n}\right) d\Gamma - \int_{\Gamma_B} \vec{N}_i \cdot \vec{n}\, \nu \operatorname{div} \vec{A}\, d\Gamma \\ &\quad - \int_{\Gamma_H} \vec{N}_i \cdot \left(\nu \operatorname{curl} \vec{A} \times \vec{n}\right) d\Gamma - \int_{\Gamma_H} \vec{N}_i \cdot \vec{n}\, \nu \operatorname{div} \vec{A}\, d\Gamma \\ &\quad = 0 \end{aligned} \tag{56}$$

Since the shape functions appertaining to the given potential values of the Dirichlet boundary conditions (45) and (48) are not used as weighting functions these functions satisfy:

$$\vec{n} \times \vec{N}_i = \vec{0} \quad \text{on } \Gamma_B \tag{57}$$

$$\vec{n} \cdot \vec{N}_i = 0 \quad \text{on } \Gamma_H \tag{58}$$

For that reason the third and sixth surface integral of (56) are zero. The other surface integral vanishes by the satisfaction of the boundary conditions (42) - (44),(46) and (47).

Finally the transformations of (54) - (56) result in the following three equations which, written for all nodes, give a linear equation system for the nodal potential values $\vec{A}$ and V of the FE mesh:

$$\int_{\Omega_1} \Big(\nu \operatorname{curl} \vec{N}_i \cdot \operatorname{curl} \vec{A} + \nu \operatorname{div} \vec{N}_i \cdot \operatorname{div} \vec{A}$$

$$- \vec{N}_i \cdot \sigma \vec{v} \times \operatorname{curl} \vec{A} + \vec{N}_i \cdot \sigma \operatorname{grad} V\Big) \, d\Omega = 0 \quad (59)$$

$$\int_{\Omega_1} \Big(\sigma \operatorname{grad} N_i \cdot \vec{v} \times \operatorname{curl} \vec{A} - \sigma \operatorname{grad} N_i \cdot \operatorname{grad} V\Big) \, d\Omega = 0 \quad (60)$$

$$\int_{\Omega_2} \Big(\nu \operatorname{curl} \vec{N}_i \cdot \operatorname{curl} \vec{A} + \nu \operatorname{div} \vec{N}_i \cdot \operatorname{div} \vec{A} - \vec{J}_s\Big) \, d\Omega = 0 \quad (61)$$

The third term in (59) representing the velocity effects forces the matrix to be badly posed and causes oscillations of the solution process. To reduce these oscillations and to achieve a faster convergence an upwind scheme like it is described in [1], [2] and [5] has been carried over to tetrahedral elements and is implemented in the third term of (59). The resulting matrix from (59) - (61) is solved by a BiCG procedure and a stabilized BiCG procedure [6].

To consider non-linear materials an underrelaxating iteration procedure is used for the calculation of the permeability μ_{n+1} for the next linear computation step dependent on the previous one μ_n and the permeability distribution μ_{res} resulting from the field solution:

$$\mu_{n+1} = \alpha \, \mu_{res} + (1 - \alpha) \, \mu_n \quad (62)$$

Compared to calculations without velocity effects the number of non-linear iteration steps is considerably higher. To achieve a faster convergence on the one hand side combined with a good accuracy on the other side, α is reduced exponentially throughout the calculation process:

$$\alpha = \alpha_0^n \quad . \quad (63)$$

The constant α_0 is determined by a series of 2D calculations. Starting with $\mu_r = 5$ and $\alpha_0 = 0.6$ in the ferromagnetic regions we obtained best results. Depending on the material and the velocity 20 to 30 non-linear steps are necessary.

III. Application and Results

Linear eddy current braking systems are used in magnetic levitation applications and in high velocity trains in order to avoid the abrasion of mechanical disc brakes.

Fig. 1 shows one pole of the calculated brake usually consisting of 6 or 12 poles in a longitudinal section view. The pole length is 180 mm and the non-linear characteristic of the B-H curve of iron is considered as well in the yoke as in the rail. Since the modeling of the whole brake would require a large computation expenditure only one pole is taken into account. The computed coarse mesh consists of about 25000 first order tetrahedral elements with 5000 nodes, the finer mesh of about 100000 elements with 20000 nodes.

Fig. 3 shows the absolute value of the computed flux density distribution for one pole in a longitudinal section view. Measurements on a train and on a test bench compared to the resulting braking forces calculated by the Maxwell-tensor method are displayed in Fig. 4.

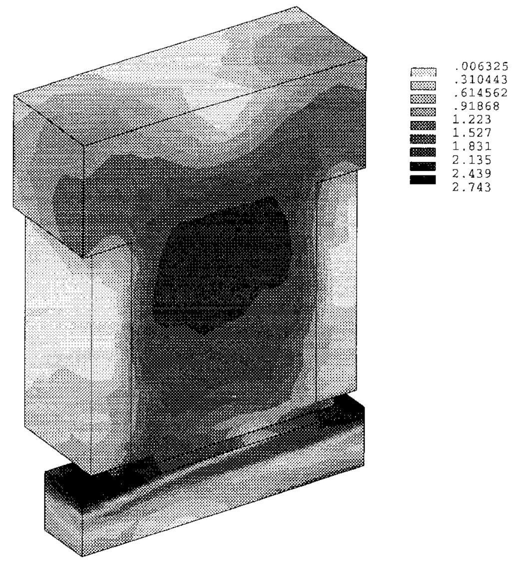

Fig. 3. Distribution of the absolute value of $\vec{B}$ (T) for 100 km/h

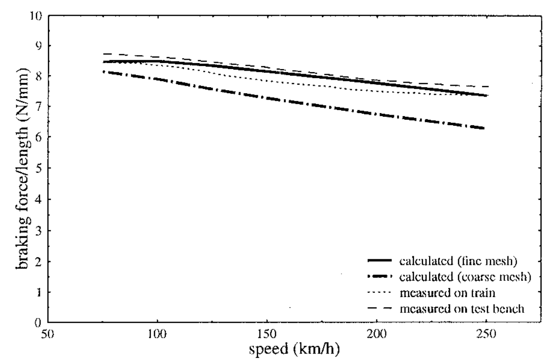

Fig. 4. Computed braking force compared to measurements

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