Biorthogonal Shape Functions for Nonconforming Sliding Interfaces in 3-D Electrical Machine FE Models With Motion

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This paper discusses the application of Lagrange multipliers to restore field continuity across nonconforming surfaces in 3-D problems. The method makes it in particular possible to implement the relative motion of stator and rotor without remeshing in the 3-D Finite Element (FE) modeling of electrical machines. The choice of a special set of biorthogonal shape functions for the Lagrange multiplier makes it possible to preserve the positive definiteness of the FE system. It is shown that such a biorthogonal basis cannot be constructed canonically for a 3-D magnetic vector potential formulation. For a 3-D magnetic scalar potential formulation, however, the situation is different and a biorthogonal basis can be found.

***Index Terms*—Biorthogonal shape functions, electric machines, finite element methods, Lagrange multiplier, sliding interfaces.**

I. INTRODUCTION

RECENT investigations [1] on the applicability of biorthogonal shape functions in the Finite Element analysis (FEA) of electrical machines have shown considerable potential towards a generic approach for 2-D and 3-D problems with motion. A standard formulation for motional problems is the moving band (MB) technique [2], which, for practical reasons, is only applicable to 2-D rotating machines. The mortar element method (MEM) based on Lagrange multiplier (LM) was discussed in [3] and applied to a 2-D machine problem in [4] and has been extensively studied in [5]. The MEM can be extended to 3-D problems, but requires the integration to be performed on an intermediate surface mesh [6] or the conditioning of the system matrix worsens significantly [7].

The nonconforming approach presented in this paper belongs to the category of LM methods, but instead of using standard basis functions for the Lagrange multiplier, the special biorthogonal basis functions proposed in [8] are used. The biorthogonality property makes it possible to eliminate algebraically the Lagrange multiplier, turning the saddle point problem (which is typical of LM approaches) into a symmetric, positive definite system of equations. However, biorthogonal edge-based Whitney functions could not be constructed in a canonical way. In 3-D, the technique can thus be applied to magnetic scalar potential $\mathbf{T}-\omega$ formulations, but not to magnetic vector potential $\mathbf{A}$ formulations.

II. VARIATIONAL FORMULATION

Let Ω^m and Ω^s be two complementary domains called master and slave, $\Omega^m \cup \Omega^s = \Omega$, e.g. the stator and rotor of an electric machine. Let $\Gamma^m \subset \partial\Omega^m$ and $\Gamma^s \subset \partial\Omega^s$ be their common interface and $p : \Gamma^s \rightarrow \Gamma^m$ be a smooth mapping, which may or not account for a relative sliding between the master and the slave domain. We shall moreover assume that, as is generally the case in the kind of applications we have in scope, the sliding interfaces Γ^m and Γ^s are connected and do not intersect a Dirichlet or anti-symmetry boundary condition. If these conditions are not fulfilled, a special treatment is required as explained in [9], which is beyond the scope of this paper.

In a magnetic scalar potential $\mathbf{T}-\omega$ formulation [10], the magnetic field $\mathbf{H}^k = \mathbf{T}^k - \mathrm{grad}\omega^k, k \in \{m, s\}$, is expressed in terms of an electric vector potential $\mathbf{T}^k$ such that $\mathbf{J}^k = \mathrm{curl}\mathbf{T}^k$ and a singled-valued scalar magnetic potential ω^k. Denoting $\Phi^k(\mathbf{H}^k)$ the magnetic coenergy of the domain k as a function of the magnetic field $\mathbf{H}^k$, the variational formulation of the problem reads

$$\delta \sum_{k=m,s} \Phi^k(\mathbf{H}^k) + \delta \int_{\Gamma^s} \lambda(\omega^s - \omega^m \circ p)\mathrm{d}\Gamma = 0.$$

The second term is an additional functional ensuring the continuity of the scalar potential across the master-slave interface. The unknown field λ is the Lagrange multiplier, and the master side scalar potential ω^m is composed with the mapping p to be compared with the slave side potential ω^s on Γ^s. Assuming $\delta\mathbf{T}^k = 0$ and

$$\mathbf{B}^k \equiv \partial_{\mathbf{H}^k}\Phi^k = \bar{\mu}(\mathbf{H}^k)(\mathbf{T}^k - \mathrm{grad}\,\omega^k) \quad (1)$$

the weak formulation is

$$\sum_{k=m,s} \int_{\Omega^k} \mathbf{B}^k \cdot \mathrm{grad}\,\delta\omega^k \mathrm{d}\Omega + \int_{\Gamma^s} \delta\lambda(\omega^s - \omega^m \circ p)\mathrm{d}\Gamma + \int_{\Gamma^s} \lambda(\delta\omega^s - \delta\omega^m \circ p)\mathrm{d}\Gamma = 0 \quad (2)$$

which must be verified for all $\delta\omega^k$ and $\delta\lambda$ fulfilling the homogeneous boundary conditions.

III. EULER-LAGRANGE EQUATIONS

The variables ω^m, ω^s and λ, being linearly independent, their variations are also linearly independent. After an integration by parts, the Euler-Lagrange equations of (2) are

$$\mathrm{div}\mathbf{B}^k = 0 \qquad \text{in } \Omega^k \text{ with } k \in \{m, s\} \quad (3)$$
$$\mathbf{n}^s \cdot \mathbf{B}^s = \lambda \qquad \text{on } \Gamma^s \quad (4)$$

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$$\mathbf{n}^m \cdot \mathbf{B}^m = -\lambda \circ p^{-1} \qquad \text{on } \Gamma^m = p\Gamma^s \tag{5}$$
$$\omega^s = \omega^m \circ p \qquad \text{on } \Gamma^s. \tag{6}$$

Equation (3) is Gauss law. Equations (4) and (5) indicate the physical interpretation of the Lagrange multiplier λ as the magnetic flux across the sliding interfaces Γ. The continuity of the magnetic scalar potential ω is ensured by (6).

IV. Discrete Formulation

In order to establish the FE equations in matrix form, the vectors of unknowns $\mathbf{u}^k$, $k \in \{m, s\}$ are divided into two blocks each (see also [1])

$$\mathbf{u}^m = \begin{pmatrix} \mathbf{u}_i^m \\ \mathbf{u}_\Gamma^m \end{pmatrix}, \quad \mathbf{u}^s = \begin{pmatrix} \mathbf{u}_\Gamma^s \\ \mathbf{u}_i^s \end{pmatrix}. \tag{7}$$

The block $\mathbf{u}_\Gamma^k$ contains the unknowns lying on the sliding interfaces Γ^k, whereas the block $\mathbf{u}_i^k$ contains the unknowns lying in the interior of the domains Ω^k. The magnetic scalar potential ω and λ are both discretized with nodal shape functions

$$\omega^k = \sum_l \alpha_l \omega_l^k, \qquad \delta\omega^k = \{\alpha_l^k\} \tag{8}$$
$$\lambda = \sum_l \mu_l \lambda_l, \qquad \delta\lambda = \{\mu_l\}. \tag{9}$$

The superscript ^{s} is omitted for μ because the shape functions of λ are defined on Γ^s only. Equation (2) yields then the saddle-point problem

$$\begin{pmatrix} \mathbf{S}_{i,i}^m & \mathbf{S}_{i,\Gamma}^m & 0 & 0 & 0 \\ \mathbf{S}_{\Gamma,i}^m & \mathbf{S}_{\Gamma,\Gamma}^m & 0 & 0 & -\mathbf{M}^T \\ 0 & 0 & \mathbf{S}_{\Gamma,\Gamma}^s & \mathbf{S}_{\Gamma,i}^s & \mathbf{D}^T \\ 0 & 0 & \mathbf{S}_{i,\Gamma}^s & \mathbf{S}_{i,i}^s & 0 \\ 0 & -\mathbf{M} & \mathbf{D} & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u}_i^m \\ \mathbf{u}_\Gamma^m \\ \mathbf{u}_\Gamma^s \\ \mathbf{u}_i^s \\ \boldsymbol{\lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{b}^m \\ 0 \\ 0 \\ \mathbf{b}^s \\ 0 \end{pmatrix} \tag{10}$$

with

$$S_{ln}^k = \int_{\Omega^k} \bar{\mu} \operatorname{grad} \alpha_l^k \cdot \operatorname{grad} \alpha_n^k \mathrm{d}\Omega \tag{11}$$
$$b_l^k = \int_{\Omega^k} \mathbf{T}^k \cdot \operatorname{grad} \alpha_l^k \mathrm{d}\Omega \tag{12}$$
$$D_{jl} = \int_{\Gamma^s} \mu_j \alpha_l^s \mathrm{d}\Gamma, \quad M_{jl} = \int_{\Gamma^s} \mu_j \alpha_l^m \circ p \mathrm{d}\Gamma. \tag{13}$$

In order to obtain a symmetric positive definite system, the degrees of freedom $\mathbf{u}_\Gamma^s$ associated to the slave side Γ^s of the sliding interface are eliminated thanks to the last block-line of the saddle-point system (10) and expressed by a linear combination of $\mathbf{u}_\Gamma^m$

$$\mathbf{D}\mathbf{u}_\Gamma^s - \mathbf{M}\mathbf{u}_\Gamma^m = 0 \tag{14}$$

so that

$$\mathbf{u}_\Gamma^s = \mathbf{D}^{-1}\mathbf{M}\mathbf{u}_\Gamma^m \equiv \mathbf{Q}\mathbf{u}_\Gamma^m \tag{15}$$

where the rectangular matrix

$$\mathbf{Q} \equiv \mathbf{D}^{-1}\mathbf{M} \equiv (\mathbf{M}^T\mathbf{D}^{-T})^T \tag{16}$$

plays the role of a discrete master-to-slave continuity operator. The Lagrange multiplier, on the other hand, can be extracted from the third line of (10) and (15)

$$\boldsymbol{\lambda} = -\mathbf{D}^{-T}\mathbf{S}_{\Gamma,\Gamma}^s\mathbf{D}^{-1}\mathbf{M}\mathbf{u}_\Gamma^m - \mathbf{D}^{-T}\mathbf{S}_{\Gamma,i}^s\mathbf{u}_i^s \tag{17}$$

and substituting (15) and (17) into (10) yields

$$\begin{pmatrix} \mathbf{S}_{i,i}^m & \mathbf{S}_{i,\Gamma}^m & 0 \\ \mathbf{S}_{\Gamma,i}^m & \mathbf{S}_{\Gamma,\Gamma}^m + \mathbf{Q}^T\mathbf{S}_{\Gamma,\Gamma}^s\mathbf{Q} & \mathbf{Q}^T\mathbf{S}_{\Gamma,i}^s \\ 0 & \mathbf{S}_{i,\Gamma}^s\mathbf{Q} & \mathbf{S}_{i,i}^s \end{pmatrix} \begin{pmatrix} \mathbf{u}_i^m \\ \mathbf{u}_\Gamma^m \\ \mathbf{u}_i^s \end{pmatrix} = \begin{pmatrix} \mathbf{b}^m \\ 0 \\ \mathbf{b}^s \end{pmatrix}. \tag{18}$$

This system of equations is symmetric, positive definite and can, contrary to (10), be solved efficiently by standard Krylov subspace methods. However, to obtain (18) it is necessary to evaluate $\mathbf{D}^{-1}$, as seen in (16). The structure of $\mathbf{D}$ depends on the choice of the shape functions for the Lagrange multiplier λ. We now show that an appropriate choice of the basis functions allows diagonalizing $\mathbf{D}$ so that its inversion becomes trivial, and the relationship between master and slave degrees of freedom can be handled element by element during the assembly, without having to deal with a fully populated inverse matrix $\mathbf{D}^{-1}$ in (16).

V. Biorthogonal Shape Functions

If the matrix $\mathbf{D}$ is diagonal, the evaluation of (16) reduces to a simple matrix product. This is the case when the shape functions of the Lagrange multiplier verify the biorthogonality relation [8], [9]

$$D_{jl} = \int_{\Gamma^s} \mu_j \alpha_l^s \mathrm{d}\Gamma = \delta_{jl} \int_{\Gamma^s} \alpha_l^s \mathrm{d}\Gamma, \quad \text{with} \quad \delta_{jl} = \begin{cases} 1 & \text{if } j = l, \\ 0 & \text{if } j \neq l \end{cases} \tag{19}$$

Several families of functions, continuous or not, fulfill a biorthogonality condition. The biorthogonality relation (19) involves a 2-D integral on the sliding interface Γ^s. Biorthogonal nodal shape functions μ_k with $k = 1, 2, 3$ associated with the node p_k can be found as functions of the corresponding barycentric coordinates ξ_k (cf. [11])

$$\mu_k = -30\xi_k^3 + 45\xi_k^2 - 15\xi_k - \xi_{(k+1)modulo3} - \xi_{(k+2)modulo3} + 30\xi_1\xi_2\xi_3 + 1. \tag{20}$$

They are of third polynomial order, which is necessary to ensure continuity at the vertices, and depicted in Fig. 1. Keep in mind, that by definition of the barycentric coordinates one has the identity $\xi_3 = 1 - \xi_1 - \xi_2$.

VI. Magnetic Vector Potential

In case of a magnetic vector potential formulation, the unknown vector field $\mathbf{A}$ is discretized with edge-based Whitney

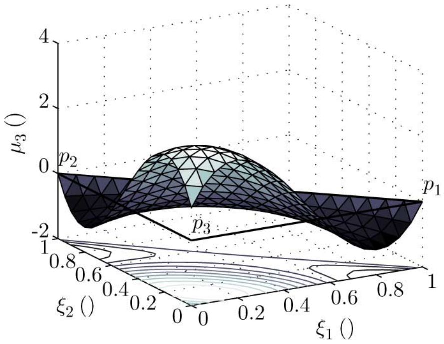

Fig. 1. Biorthogonal nodal shape functions for the 3-D case.

shape functions $\{\boldsymbol{\alpha}_l\}$. On the other hand, the Lagrange multiplier is also a vector field in this case, which must hence be discretized with edge-based shape functions $\{\boldsymbol{\mu}_l\}$ as well. One has thus

$$\mathbf{A}^k = \sum_l \boldsymbol{\alpha}_l \mathbf{A}_l^k, \qquad \delta\mathbf{A}^k = \{\boldsymbol{\alpha}_l^k\} \tag{21}$$

$$\boldsymbol{\lambda} = \sum_l \boldsymbol{\mu}_l \boldsymbol{\lambda}_l, \qquad \delta\boldsymbol{\lambda} = \{\boldsymbol{\mu}_l\}. \tag{22}$$

In order to apply the proposed approach, the vector shape functions $\{\boldsymbol{\mu}_l\}$ should now be chosen such that a biorthogonality relationship

$$D_{jl} = \int_{\Gamma^s} \left(\boldsymbol{\mu}_j \times \boldsymbol{\alpha}_l^s\right) \cdot \mathbf{n}^s \mathrm{d}\Gamma = \delta_{jl} C_l \tag{23}$$

holds, with C_l a normalizing factor depending on $\boldsymbol{\alpha}_l^s$. However, due to the incompatibility between the antisymmetry of the vector product at the left-hand side of (23) and the symmetry of the Kronecker symbol δ_{jl} at the right-hand side, this turns out to be impossible.

VII. Example Applications

Two benchmark problems are thus presented with scalar unknown fields: an academic thermal problem with two nested bricks and a permanent magnet excited synchronous motor. The formulations have been implemented within the FEM-package iMOOSE [12].

A. Heat Equation Problem

The thermal problem consists of two concentric bricks, as depicted in Fig. 2, and the nonconforming formulation consists of the terms (11) and (13). The small ends of the outer brick are subject to inhomogeneous Dirichlet boundary conditions whereas the side surfaces are subject to homogeneous Neumann boundary conditions. The interfaces Γ^k between the nested bricks are meshed nonconforming (see Fig. 3). Numerical experiments show that, even for significantly different mesh sizes on the slave and master interfaces Γ^s and Γ^m, the continuity of the field is satisfactorily preserved. Swapping the master and slave sides, on the other hand, does not lead to an increase of the error, but a slow down of the convergence rate of the CG-algorithm is observed.

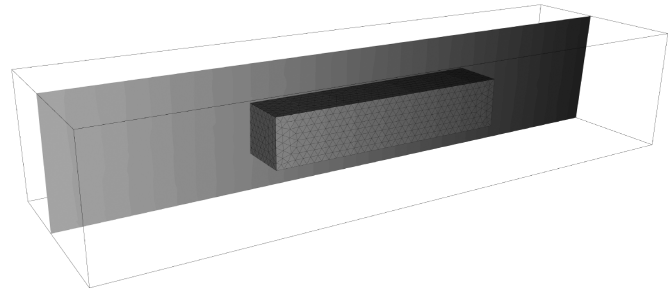

Fig. 2. Solution of Fourier's heat equation on nonconforming discretization on Γ of two nested bricks.

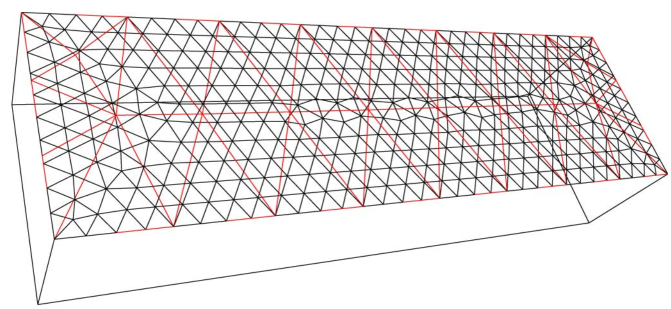

Fig. 3. Nonconforming discretization on surface Γ^s of the inner brick.

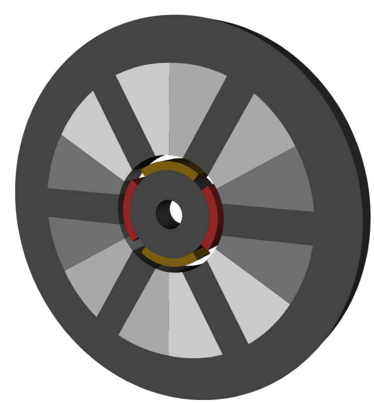

Fig. 4. Geometry of the inspected synchronous machine.

B. Permanent Magnet Excited Synchronous Machine

The permanent magnet excited synchronous motor depicted in Fig. 4 is a more challenging application. The 3-D model is extruded from a 2-D geometry, so that a reference solution is available. The sliding interface is a concentric cylinder in the air gap with nonconforming discretization on the master and slave sides, as can be seen in Fig. 7. The source field $\mathbf{T}^k$ of the $\mathbf{T}-\omega$ formulation is determined by the permanent magnets in this application [13]. The proposed approach would however work the same way with a coil system. Neumann boundary conditions (no flux) apply on all external surfaces of the model. The sliding interfaces Γ^m and Γ^s intersect thus the Neumann boundary surface at the front and back ends of the model in axial direction.

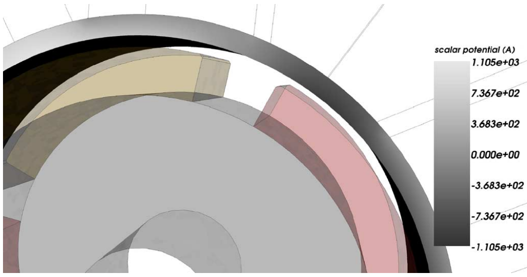

Fig. 5. Magnetic scalar potential ω around the interface Γ.

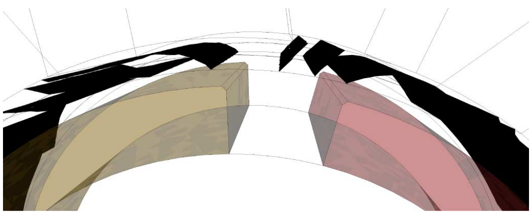

Fig. 6. Isoplanes of magnetic scalar potential ω across the interface Γ.

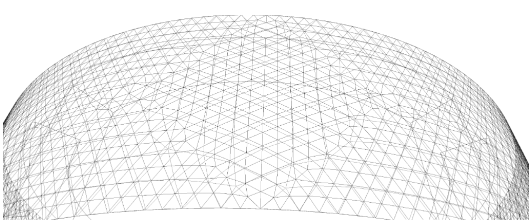

Fig. 7. Nonconforming discretization on the sliding interface in the air gap.

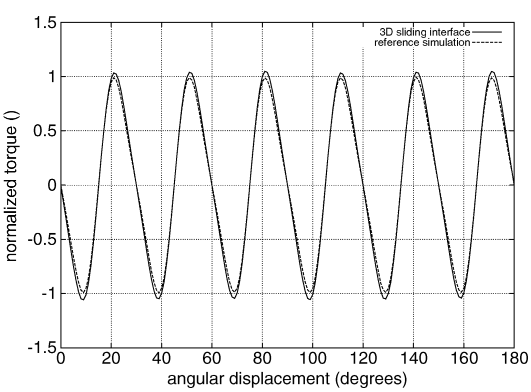

Fig. 8. Comparison of resulting cogging torque with reference simulation.

In Fig. 5, the computed magnetic scalar potential ω is represented in the vicinity of the sliding interface Γ. The restored continuity of the potential ω is controlled in Fig. 6 by plotting isovalue surfaces of ω and checking that they are continuous across the nonconformally discretized interface Γ. The cogging torque has been calculated and is compared with the 2-D reference solution in Fig. 8. It shows a very good agreement. Finally, it is worth mentioning that a periodic boundary condition, instead of a Neumann boundary condition, can be implemented as well, provided due care is taken at the intersection with the sliding interface.

VIII. Conclusion

In this work, Lagrange multipliers are used to ensure the continuity of a scalar field across a nonconforming interface between two separated domains, e.g. the stator and the rotor of an electrical machine. By choosing for the discrete Lagrange multiplier field basis functions that fulfill a biorthogonality relation, the Lagrange multiplier unknowns can be eliminated algebraically. An indefinite saddle point problem is avoided this way, and a symmetric positive definite system of equations is obtained, making it possible to solve the problem efficiently with standard iterative solvers. It has been shown however that a biorthogonal basis cannot be found for edge-based unknown fields, e.g. for magnetic vector potential formulations. Still the method has been applied successfully to a 3-D thermal problem and a permanent magnet excited synchronous motor with a magnetic scalar potential formulation. The proposed method makes it possible to solve efficiently motional 3-D FE electrical machine problems without remeshing and generalizes the 2-D method that was proposed in [1]. The research now focuses on proving the efficiency of the biorthogonal Lagrange multiplier method in the case of eddy current problems.

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