Upwind 3-D Vector Potential Formulation for Electromagnetic Braking Simulations

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The calculation of motion-induced eddy currents in massive conductors yields a 3-D convection-diffusion problem. Up-winding and SUPG formulations are well established methods to obtain stable discretizations of the scalar convection-diffusion equations in the case of singular perturbation, but there is very little reported experience with the stability of convection in the vector case, i.e., electromagnetism. Numerical experiments with the up-winding method proposed by Xu *et al.* (Trans. on Mag., 2006; 42:667–670, 2006) has proven it to be insufficient. Building on the work of Heumann *et al.* (Research report 2008-30, Seminar für Angewandte Mathematik, Eidgenssische Technische Hochschule, Oct. 2008), an alternative approach based on a finite-element discretization of the Lie derivative implied by the convection phenomenon is proposed.

***Index Terms*—Finite elements, lie derivative, motion-induced eddy currents, upwinding, Whitney elements.**

I. Weak Formulation

MAGNETODYNAMICS problems can be solved with the vector magnetic potential a and the electric scalar potential u as unknown fields. The weak formulation is obtained by orthogonalizing the governing equations with the appropriate test function spaces in accordance with the finite-element theory

$$\int_\Omega [\operatorname{curl} \bar{h} - j] \cdot a' \, d\Omega + \int_\Omega [\operatorname{div} j] u' \, d\Omega = 0 \quad \forall a', u' \tag{1}$$

where $\bar{h} \equiv \nu \operatorname{curl} a$, and the current density j is expressed in terms of the unknown fields as

$$j = -\sigma[D_t a + \operatorname{grad} u]. \tag{2}$$

The operator D_t is the co-moving time derivative, which is also commonly called material derivative or total derivative in different contexts. The co-moving time derivative of the magnetic vector potential a writes

$$\begin{aligned} D_t a &= \partial_t a + \operatorname{grad}(a \cdot \mathbf{v}) - \mathbf{v} \times \operatorname{curl} a & (3)\\ &= \partial_t a + \mathrm{di}_\mathbf{v} a + \mathrm{i}_\mathbf{v} \mathrm{d} a = \partial_t a + \mathcal{L}_\mathbf{v}\, a & (4)\end{aligned}$$

respectively, terms of vector analysis and differential geometry quantities and operators (see, e.g., [3] for an expository monography and [4] for a reference book). Term by term indentifications define the vector analysis equivalents of differential geometry operators that play an important role in this discussion: namely the exterior derivative d, the inner derivative $i_\mathbf{v}$ and the Lie derivative $\mathcal{L}_\mathbf{v}$. In particular, given a placement map $p_t : M \mapsto \Omega$, the geometrical interpretation of the co-moving time derivative is that it is the derivative that fulfills

$$\partial_t \int_\Omega a = \int_\Omega D_t a$$

which implies, with p_t^* the pullback of p_t

$$D_t \equiv p_t^{-*} \partial_t p_t^*$$

since

$$\begin{aligned}\partial_t \int_{\Omega = p_t M} a &= \partial_t \int_M p_t^* a \\ &= \int_{M = p_t^{-1}\Omega} \partial_t p_t^* a = \int_\Omega p_t^{-*} \partial_t p_t^* a.\end{aligned}$$

In the stationary case, $\partial_t a$ is zero, and $D_t = \mathcal{L}_\mathbf{v}$. Substituting (2) in (1), the weak formulation of the problem is obtained

$$\begin{aligned}&\int_\Omega [\operatorname{curl} \bar{h} + \sigma \mathcal{L}_\mathbf{v}\, a + \sigma \operatorname{grad} u] \cdot a' \, d\Omega \\ &+ \int_\Omega \sigma[\mathcal{L}_\mathbf{v}\, a + \operatorname{grad} u] \cdot \operatorname{grad} u' \, d\Omega \\ &+ \int_{\partial\Omega} \bar{h} \cdot a' \, d\partial\Omega - \int_{\partial\Omega} \sigma[\mathcal{L}_\mathbf{v}\, a + \operatorname{grad} u]\, u' \, d\partial\Omega = 0 \end{aligned} \tag{5}$$

$\forall a', u'$, as it is used by Xu *et al.* with $a' \equiv \vec{W}$ and $u' \equiv w$ [1]. Note that they assume $\mathcal{L}_\mathbf{v}\, a = -\mathbf{v} \times \operatorname{curl} a$, which amounts to using a modified scalar electric potential $\tilde{u} = u + a \cdot \mathbf{v}$.

II. Upwinding in 3-D Simplicial Meshes

A. Continuity

The tangential component of a and $\partial_t a$ is continuous across finite elements surfaces. One has

$$a = \sum_i A_i \omega_i^e(\mathbf{x}), \quad A_i = \int_{e_i} a \Big|_{\Omega_i} \tag{6}$$

where $\omega_i^e(\mathbf{x})$ are Whitney edge shape functions, and where the edge connector A_i can be evaluated by integration of the discretized field $a|_{\Omega_i}$ of *any* tetrahedron Ω_i adjacent to the edge e_i with the same result. For $\mathcal{L}_\mathbf{v}\, a$, one has on the contrary

$$\mathcal{L}_\mathbf{v}\, a = \sum_i \left\{ \int_{e_i} \mathcal{L}_\mathbf{v}\, a \Big|_{\Omega_i} \right\} \omega_i^e(\mathbf{x})$$

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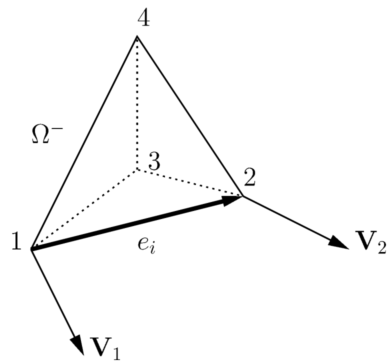

Fig. 1. Edge $e_i \equiv [n_1, n_2]$ and the upwind element Ω^-.

but the bracketed coefficient depends now on the chosen element Ω_i. There is, however, no indetermination, for the Lie derivative is by definition a *one-sided limit*

$$\begin{aligned} \int_{e_i} \mathcal{L}_{\mathbf{v}}\, a &\equiv \lim_{\Delta t \to 0+} \frac{1}{\Delta t} \left\{ \int_{e_i} p^*_{t+\Delta t} a \quad - \int_{e_i} p^*_t a \right\} \\ &= \lim_{\Delta t \to 0+} \frac{1}{\Delta t} \left\{ \int_{p_{t+\Delta t} e_i} a \quad - \int_{p_t e_i} a \right\} \end{aligned} \tag{7}$$

involving the upwinded edge $p_{t+\Delta t} e_i$. This imposes to evaluate the circulation in the upwind region relative to the edge e_i under consideration, Fig. 1. The notion of upwind region, and in particular the notion of upwind element Ω^- with respect to a node, an edge or a facet, is now made precise.

B. Hyperfaces, Orientation and Boundary Operator

All calculations done in the sequel are affine invariant. They are done in the real finite element with global orientation, i.e., the reference element is not used. Let Ω be an arbitrary simplicial element of order m in a mesh covering a m-dimensional space. Without loss of generality, we assume that this element is represented by a set of $m+1$ nodes $t = [n_1, \dots, n_{m+1}]$, whose positions in space are given by $\mathbf{x}_k$, $k = 1, \dots, m+1$. Ordered subsets of cardinality $p+1$ of t are called p-hyperfaces. The $m+1$ subsets of cardinality 1 are the vertices (0-hyperfaces) of the element, the ordered subsets of cardinality 2 are the edges (1-hyperfaces), the ordered subsets of cardinality 3 are the facets (2-hyperfaces), etc. A permutation of two vertices in a p-hyperface amounts to reverting its orientation. The boundary of a p-hyperface h is the $(p-1)$-hyperface defined as

$$\partial h = \sum_{n_k \in h} (-1)^{\pi(h, n_k) - 1} (h / n_k)$$

where h/g denotes the set obtained by removing the nodes of $g \subset h$ from h, and where $\pi(h, n_k) \in [1, p+1] \subset N$ denotes the position of n_k in the ordered set h. For instance

$$\partial [n_i, n_j] = [n_j] - [n_i]$$

and

$$\partial [n_i, n_j, n_k] = [n_j, n_k] - [n_i, n_k] + [n_i, n_j].$$

C. Simplicial Extrusion Basis

Let $h = [\dots, n_k, \dots]$ be an arbitrary p-hyperface adjacent to node n_k. We call simplicial extrusion basis of h at node n_k the set of $(p+1)$-hyperfaces obtained by making the substitution

$$\begin{aligned} h = [\dots, n_k, \dots] &\rightarrow \mathrm{SEB}(h, n_k, t) \\ &\equiv [n_k, \dots, t/h, \dots] \\ &= [[n_k, \dots, n_l, \dots]] \end{aligned} \tag{8}$$

with $n_l \in t/h$, where the double brackets recalls that the result is a set of hyperfaces. The cardinality of this set is $m - p$.

D. Simplicial Basis for the Tangent Space

Considering a particular simplicial element Ω, represented equivalently by its vertice set t, the nodal values of the velocity field are denoted by $\mathbf{v}_k$, $k = 1, \dots, m+1$, and the discretized velocity field writes

$$\mathbf{v}(\mathbf{x}) = \sum_{k=1}^{m+1} \mathbf{v}_k \omega_k^{n,t}(\mathbf{x}), \quad \mathbf{v}_k = \sum_{l=1}^{m} V_{kl}^{\mathrm{Cart}} \mathbf{E}_l^{\mathrm{Cart}} \tag{9}$$

where the $\omega_k^{n,t}(\mathbf{x})$'s are the nodal barycentric shape functions of the element and the vectors $\mathbf{E}_l^{\mathrm{Cart}}$ form a Cartesian global reference frame for the m-dimensional space.

The simplicial extrusion basis of a node $[n_k]$ is the set

$$\mathrm{SEB}(n_k, n_k, t) = [n_k, t/n_k] = [[n_k, n_l]], \quad n_l \in t/n_k$$

of m edges adjacent to node n_k and oriented outwards. This set is an alternative basis for vectors anchored at n_k. We adopt the notation

$$\left[E_{kl}^t\right] := \mathrm{SEB}(n_k, n_k, t) = [[n_k, n_l]], \quad n_l \in t/n_k$$

so that the discretized velocity field writes in this basis

$$\mathbf{v}(\mathbf{x}) = \sum_{k=1}^{m+1} \mathbf{v}_k \omega_k^{n,t}(\mathbf{x}), \quad \mathbf{v}_k = \sum_{n_l \in t - n_k} V_{kl}^t E_{kl}^t \tag{10}$$

where the edges $E_{kl}^t = [n_k, n_l]$, which are (straight) curves, are assimilated with their tangent vector at n_k to form a basis for vectors in $T_{n_k}(\Omega)$. The coefficients of this expansion are given by

$$V_{kl}^t = \omega_l^{n,t}(\mathbf{x}_k + \mathbf{v}_k). \tag{11}$$

Note that this evaluation may occur outside the volume enclosed by the simplicial element Ω. Barycentric shape functions are defined over $\mathbb{R}^m$. The superscript t recalls throughout which quantities are simplex-dependent.

E. Implied Orientation

Given an orientation of the p-hyperface h, the elements of the simplicial extrusion basis $\mathrm{SEB}(h, n_k, t)$ acquire by virtue of the definition (8) an implied orientation.

The simplicial extrusion basis of a node $[n_k]$, for instance, is a set of m edges adjacent to node n_k and oriented outwards. Hence, because of the natural outwards orientation, the boundary of those edges contains the initial node n_k with a negative coefficient. The definition (8) is such that this rule is generalized towards higher degrees as well. The boundary of each element of the simplicial extrusion basis $\mathrm{SEB}(h, n_k, t)$ contains h with a negative sign, i.e.,

$$\tilde{t} \in \mathrm{SEB}(h, n_k, t) \rightarrow -h \in \partial\tilde{t}.$$

F. Upwind Element

Equation (8) shows that each of the $m-p$ elements of the simplicial extrusion basis $\mathrm{SEB}(h, n_k, t)$ can be associated with a node $n_l \in t/h \subset t/n_k$ and, therefore, with a coefficient V^t_{kl} of the local velocity field (10).

The upwind element Ω^-, or t^-, of a pair (h, n_k), where h is a p-hyperface adjacent to the node n_k, is by definition the element adjacent to h, i.e., $h \subset t^-$, for which the $m-p$ conditions

$$V^{t^-}_{kl} \leq 0, \quad n_l \in t^-/h \tag{12}$$

hold. This is an unambiguous definition; there is exactly one upwind element for each pair (h, n_k).

There are thus up to $p+1$ upwind elements for a p-hyperface, one for each node of h. In particular, there exists exactly one upwind element for a node n_k (0-hyperface), but the upwind elements $\Omega^-(h, n_i)$ and $\Omega^-(h, n_j)$ of an edge $h = [n_i, n_j]$ (1-hyperface) may differ. There are also, in principle, three different upwind elements for a facet (2-hyperface), one for each node, but, for there are at most two elements adjacent to a facet in a 3-D space ($m = 3$), the number of upwind elements of a facet is also at most two in 3-D.

Fig. 2 (top) shows the upwind element a node $n_1 \in \Omega$. This element, noted $\Omega^-(n_1, n_1)$, needs have at least node n_1 in common with Ω. It can also share an edge, a facet or even be identical to Ω according to the direction of the velocity $\mathbf{v}_1$. Fig. 2 (bottom) shows the upwind elements of an edge $e_i = [n_1, n_2]$. In this case, the velocity field is such that the upwind elements $\Omega^-(e_i, n_1)$ and $\Omega^-(e_i, n_2)$ are distinct. This happens because the upwind extrusion of n_1 (backwards prolongation of $\mathbf{v}_1$) is in between the half-planes defined by the nodes $\{n_1, n_2, n_3\}$ and $\{n_1, n_2, n_4\}$, whereas the upwind extrusion of n_2 (backwards prolongation of $\mathbf{v}_2$) is in between the half-planes defined by the nodes $\{n_1, n_2, n_3\}$ and $\{n_1, n_2, n_5\}$. The elements $\Omega^-(e_i, n_1)$ and $\Omega^-(e_i, n_2)$ need have at least the edge e_i in common, but they may share a facet or be identical.

G. Extrusion

The auxiliary notion of simplicial extrusion basis defined above allows a straightforward expression of the extrusion operator defined by A. Bossavit [5]. Combining both definitions in (10), the velocity field over $\Omega \equiv t$ reads

$$\begin{aligned}\mathbf{v}(\mathbf{x}) &= \sum_{k=1}^{m+1} \sum_{n_l \in t/n_k} V^t_{kl}\, E^t_{kl} \omega^{n,t}_k(\mathbf{x}) \\ &\equiv \sum_{k=1}^{m+1} \sum_{n_l \in t/n_k} V^t_{kl} \omega^t_{kl}(\mathbf{x})\end{aligned}$$

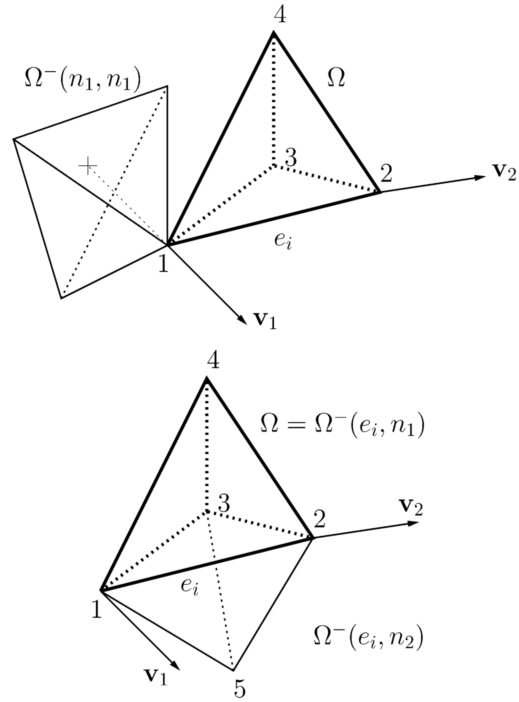

Fig. 2. Upwind elements for one node (top) and for one edge (bottom).

i.e., a double sum expansion in terms of simplicial-dependent scalar coefficients and vector basis functions. Each of these basis functions

$$\omega^t_{kl}(\mathbf{x}) \equiv E^t_{kl}\, \omega^{n,t}_k(\mathbf{x}) = [n_k, n_l]\, \omega^{n,t}_k(\mathbf{x}),\ n_l \in t/n_k \tag{13}$$

is a velocity fields defined on Ω, whose value is zero at all nodes but n_k and whose value at n_k is oriented along one edge adjacent to n_k. Those basis velocity fields have the remarkable property that they map by extrusion (with $\tau = 1$) the simplicial structure onto itself (nodes of the mesh are mapped on nodes of the mesh, edges onto edges, etc.). More precisely, they realize the vertice operation (8). The extrusion of an hyperface h adjacent to the node n_k by the set $[\omega^t_{kl}(\mathbf{x})]$, $n_l \in t/h$ of all velocity basis function associated with that node, is indeed

$$\mathrm{Ext}_\tau([\omega^t_{kl}(\mathbf{x})], h) = \tau\, \mathrm{SEB}(n_k, h, t),\ \tau \in [0, 1] \subset \mathbb{R}. \tag{14}$$

For $\tau = 1$, the result is $\mathrm{SEB}(n_k, h, t)$, which is a set of subsets of t. By construction, the extrusion operator is linear in τ, for $\tau \leq 1$. One speaks here of linearity in the formal vector space of chains.

By linearity with respect to the velocity argument now, the extrusion by the full velocity field on the simplicial element is expressed in the natural local basis (14) as

$$\mathrm{Ext}_\tau(\mathbf{v}(\mathbf{x}), h) = \sum_{n_k \in h} \sum_{n_l \in t^-/h} \tau V^{t^-}_{kl}\ \mathrm{Ext}_1\left(\omega^{t^-}_{kl}(\mathbf{x}), h\right) \tag{15}$$

where the upwinding condition (12) holds for the selection of the upwind element $t^-(h, n_k)$ associated with each pair (h, n_k). The local basis $[\omega^{t^-}_{kl}(\mathbf{x})]$ and the coefficients $[V^{t^-}_{kl}]$ are then evaluated in that element. In (15), the sum over all vertices of the simplex (first sum) can be limited to a sum over the vertices $n_k \in h$, because $\omega^{t^-}_{jl}(\mathbf{x})$ contains the nodal shape function of node n_l, see (14), which is identically zero on h if $n_j \notin h$. On

TABLE I
NUMERICAL EXPERIMENT WITH THE DISCRETIZED LIE DERIVATIVE OPERATOR

number DOFs	meshwidth	L^2-norm	rate	$H(\mathrm{curl})$-semi-norm	rate
19	1.73205	0.155442		0.87841	
98	0.866025	0.189122	−0.282942	0.937102	−0.0933103
604	0.433013	0.972814	−2.36284	0.453607	1.04676
4184	0.216506	0.68924	0.497156	0.220106	1.04325
31024	0.108253	0.460769	0.580963	0.105372	1.06271
238688	0.0541266	0.279335	0.722047	0.0519656	1.01986
1872064	0.0270633	0.179885	0.634922	0.0269615	0.946655

the other hand, the second sum is limited to the nodes $n_l \in t$ not belonging to h, because the extrusion by a vector $\mathbf{w}$ such that $i_{\mathbf{w}} h = 0$ (which would be the case if $n_l \in h$) is zero.

This leads to a discrete version of the extrusion operator, and in turn of the Lie derivative in closed form. Clearly

$$\mathrm{Ext}_\tau(\mathbf{v}, h) = \tau\, \mathrm{Ext}_1(\mathbf{v}, h)$$

so that

$$\int_h i_{\mathbf{v}} a = \lim_{\tau \to 0^-} \frac{1}{\tau} \int_{\mathrm{Ext}_\tau(\mathbf{v},h)} a = \int_{\mathrm{Ext}_1(\mathbf{v},h)} a$$

because of the linearity of chains. One ends up now with the summarizing general relation

$$\begin{aligned}\mathrm{Ext}_1(\mathbf{v}, [\ldots, n_k, \ldots]) \\ = \sum_{n_l \in t^-(h,n_k)/h} \omega_l^{n,t^-}(\mathbf{x}_k + \mathbf{v}_k)\, [n_k, \ldots, n_l, \ldots]. \quad (16)\end{aligned}$$

H. Discrete Lie Derivative

The discrete Lie derivative

$$\int_{e_i} \mathcal{L}_{\mathbf{v}}\, a = \int_{e_i} (d i_{\mathbf{v}} a + i_{\mathbf{v}} da) \equiv \int_{\Delta e_i} a \qquad (17)$$

is thus reduced to the integration of a over the 1-chain

$$\Delta e_i \equiv \partial\, \mathrm{Ext}_1(\mathbf{v}, e_i) + \mathrm{Ext}_1(\mathbf{v}, \partial e_i)$$

or, with $e_i = [n_i, n_j]$

$$\begin{aligned}\Delta[n_i, n_j] \equiv \partial\, \mathrm{Ext}_1(\mathbf{v}, [n_i, n_j]) \\ -\mathrm{Ext}_1(\mathbf{v}, n_i) + \mathrm{Ext}_1(\mathbf{v}, n_j) \quad (18)\end{aligned}$$

where the discrete operator Ext_1 is defined by (16). This 1-chain is a linear combination of edges of the mesh with velocity dependent coefficients. The final result is thus a linear combination of edge circulation of the vector potential a. This can be put into the matrix form

$$\int_{e_i} D_t a = \partial_t A_i + \sum_j Q_{ij}(\mathbf{v})\, A_j \qquad (19)$$

where the square sparse tensor $Q(\mathbf{v})$ represents the discrete Lie derivative operator. If the velocity field is constant, the matrix $Q(\mathbf{v})$ can be computed once for all.

III. NUMERICAL VALIDATION

We consider the following problem:

$$\mathrm{curl}\,\mathrm{curl}\, a - \mathbf{v} \times \mathrm{curl} a + \mathrm{grad}\,(\mathbf{v} \cdot a) + a = f \qquad (20)$$

on the unit cube $[0,1]^3$, with velocity

$$\mathbf{v} = (x - x^2)(y - y^2)(z - z^2) \begin{pmatrix} 1 \\ 0.66 \\ 0.33 \end{pmatrix}. \qquad (21)$$

We choose in these experiments the right-hand side f and boundary data such that the exact solution is

$$a = \begin{pmatrix} (x - x^2)\sin(\pi y)\sin(\pi z) \\ \sin(\pi x)(y - y^2)\sin(\pi z) \\ \sin(\pi x)\sin(\pi y)(z - z^2) \end{pmatrix}. \qquad (22)$$

The prescribed extrusion/contraction discretization is implemented within the FEniCS framework [6], and the errors in L^2-norm and $H(\mathrm{curl})$-semi-norm are calculated on a series of refined meshes. The results in Table I suggest that the error measured in the $H(\mathrm{curl})$-semi-norm converges with rate 1. We also expect this for the L^2-norm. Unfortunately, the asymptotic region is not reached in our experiments, which is however not unusual with 3-D experiments. Only a theoretical proof will be conclusive about this.

IV. CONCLUSION

This paper has presented a 3-D formulation for motion-induced currents in massive conductors. This formulation is expressed explicitly in terms of the co-moving time derivative and the Lie derivative, whose theoretical definitions clearly indicate the necessity of using an upwind discretization scheme. The convection operator has been discretized on a simplicial mesh and an upwind scheme without free parameter is obtained. Numerical experiments have shown that the proposed upwind formulation is a convergent approximation.

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